Asymptotic profiles of basic reproduction number for epidemic spreading in heterogeneous environment*

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Abstract

The effect of diffusion rates on the basic reproduction number of a general compartmental reaction-diffusion epidemic model in a heterogeneous environment is considered. It is shown when the diffusion rates tend to zero, the limit of the basic reproduction number is the maximum value of the local reproduction number on the spatial domain. On the other hand when the diffusion rates tend to infinity, the basic reproduction number tends to the spectral radius of the "average" next generation matrix. These asymptotic limits of basic reproduction number hold for a class of general spatially heterogeneous compartmental epidemic models, and they are applied to a wide variety of examples.

Keywords: Basic reproduction number; reaction-diffusion; heterogeneous environment; compartmental epidemic models.

*S. Chen is supported by National Natural Science Foundation of China (No 11771109), and J. Shi is supported by US-NSF grants DMS-1715651 and DMS-1853598.

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1 Introduction

In mathematical modeling of infectious diseases, the basic reproduction number R_0 is a key indicator for disease transmission. When $R_0 < 1$, the disease declines and eventually vanishes; and when $R_0 > 1$, the disease spreads in the population and an outbreak is possible [4]. Roughly speaking, the basic reproduction number R_0 is the average number of healthy people infected by one contagious person over the course of the infectious period. In more mathematically rigorous terms, for ordinary differential equation epidemic models which is non-spatial, R_0 is defined as the spectral radius of the next generation matrix [13, 40], which is established in a general framework of compartmental disease transmission models. This definition is also generalized to epidemic models with infinite-dimensional state space [38].

As the environment in which the disease spreads is spatially heterogeneous, the transmission and spreading of the infectious disease is inevitably affected by the spatial structure and heterogeneity of the environment. These factors can be incorporated into underlying mathematical models to show the effect of spatial heterogeneity on the disease transmission. The spatial structure and heterogeneity can be modeled in a discrete space using an ordinary differential equation patch model [1, 5, 28, 39], or they can be modeled in a continuous space using a reaction-diffusion-advection partial differential equation model [2, 11, 42, 45]. The notion of the basic reproduction number is also extended to both classes of models. In particular a theory of basic reproduction numbers for general reaction-diffusion compartmental disease transmission models is recently developed in [42].

For spatially heterogeneous reaction-diffusion epidemic models, the basic reproduction number R_0 usually depends on the diffusion rates of populations. For example, in the reaction-diffusion SIS epidemic model considered in [2]:

$$\begin{cases} \dfrac{\partial I}{\partial t} = d_I \Delta I + \beta(x)SI - \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial t} = d_S \Delta S - \beta(x)SI + \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial \nu} = \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where $\beta(x)$ is the transmission rate, $\gamma(x)$ is the removal rate, and d_I, d_S are the diffusion rates of infectious and susceptible populations respectively, it was shown that the basic reproduction number is defined as

$$R_0 = \sup \left\{ \frac{\int_\Omega \beta \phi^2 dx}{\int_\Omega (d_I |\nabla \phi|^2 + \gamma \phi^2)\, dx} : \phi \in H^1(\Omega), \phi \neq 0 \right\}. \tag{1.1}$$

Moreover it was shown in [2] that R_0 has the following asymptotic profile with respect to the infectious population diffusion rate d_I:

$$\lim_{d_I \to 0} R_0 = \max_{x \in \overline{\Omega}} \frac{\beta(x)}{\gamma(x)}, \quad \lim_{d_I \to \infty} R_0 = \frac{\int_\Omega \beta dx}{\int_\Omega \gamma dx}. \tag{1.2}$$

Notice that the quantity $\beta(x)/\gamma(x)$ is the local basic reproduction number at x when there is no spatial movement, hence the global basic reproduction number tends to the maximum of local one as the diffusion rate tends to zero. On the other hand, the limit of basic reproduction number for large diffusion rate is the ratio of average transmission rate and average removal rate. Similar asymptotic profiles for R_0 were also obtained in [31] for several kinds of other spatially heterogeneous epidemic reaction-diffusion models. The results in [31] are based on the fact that R_0 equals the spectral radius of a product of the local basic reproduction number and strongly positive compact linear operators with spectral radii one.

In this paper, we aim to characterize limiting profiles of the basic reproduction number R_0 for general spatially heterogeneous reaction-diffusion compartmental epidemic models for small or large diffusion rates.

We consider the following reaction-diffusion compartmental epidemic model

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(x, u), & x \in \Omega,\ t > 0,\ 1 \leq i \leq n, \\ \dfrac{\partial u_i}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\ 1 \leq i \leq n, \end{cases} \tag{1.3}$$

which was proposed in [42]. Here u_i is the density of the population in the i-th compartment, $d_i > 0$ is constant and represents the diffusion coefficient of population u_i, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector at $x \in \partial\Omega$, and $f_i(x, u)$ is the reaction term in the i-th compartment. Moreover,

$$f_i(x, u) = \mathcal{F}_i(x, u) - \mathcal{V}_i(x, u),$$

where $\mathcal{F}_i(x,u)$ is the input rate of newly infected individuals in the i-th compartment, $\mathcal{V}_i(x,u) = \mathcal{V}_i^-(x,u) - \mathcal{V}_i^+(x,u)$, $\mathcal{V}_i^+(x,u)$ is the rate of transfer of individuals into the i-th compartment by all other means, and $\mathcal{V}_i^-(x,u)$ is the rate of transfer of individuals out of the i-th compartment. More biological explanation of model (1.3) could be found in [42]. In this paper, we will show the asymptotic profiles of R_0 for model (1.3) as $(d_1,\cdots,d_n) \to (0,\dots,0)$ and $(d_1,\dots,d_n) \to (\infty,\dots,\infty)$. Our results indict that the trend set in [2, 31] holds true for epidemic models in much more general setting: in small diffusion limit, the global basic reproduction number tends to the maximum of local basic reproduction number, and in large diffusion limit, the global basic reproduction number tends to some kind of spatial average of local basic reproduction number.

There are extensive results on reaction-diffusion epidemic models. The asymptotic profiles of the endemic steady states were considered in [2, 33, 35, 45] and references therein, and the global dynamics of the epidemic models could be found in [8, 12, 22, 24, 26, 30, 34, 43]. The effect of diffusion and advection rates on R_0 and the stability of the disease-free steady state for a reaction-diffusion-advection epidemic model was considered in [11], see also [10, 18, 21, 32] for reaction-diffusion-advection epidemic models. The definition of R_0 for time-periodic reaction-diffusion epidemic models was given in [6, 25, 46], and the global dynamics for a time-periodic or almost space periodic reaction-diffusion SIS epidemic model was studied in [36, 41]. The reaction-diffusion epidemic models with free boundary conditions were investigated in [9, 16, 27] and references therein, and reaction-diffusion epidemic models with time delays were also studied extensively, see e.g. [7, 29, 44].

Throughout the paper, we use the following notations. For $n \ge 1$,

$$\begin{aligned} \mathbb{R}^n_+ =& \{u = (u_1,\dots,u_n) : u_i \ge 0 \ \text{ for any } \ i = 1,\dots,n\}, \\ C\left(\overline{\Omega}, \mathbb{R}^n_+\right) =& \{(u_1(x),\dots,u_n(x)) : u_i(x) \left(\in C(\overline{\Omega},\mathbb{R})\right) \ge 0 \ \text{ for any } \ i = 1,\dots,n\}. \end{aligned} \tag{1.4}$$

For a closed and linear operator A, we denote the spectral radius of A by $r(A)$, the spectral set of A by $\sigma(A)$, and the spectral bound of A by

$$s(A) := \sup\{\mathcal{R}e\lambda : \lambda \in \sigma(A)\}.$$

Let $P = (P_{ij})_{1\le i,j\le l}$ and $Q = (Q_{ij})_{1\le i,j\le l}$ be $l \times l$ ($l \ge 1$) real-valued matrices, and let $Q(x) = (Q_{ij}(x))_{1\le i,j\le l}$ be an $l \times l$ matrix-valued function.

$P \ge Q$ means $P_{ij} \ge Q_{ij}$ for each $1 \le i, j \le l$.

$P > Q$ means $P_{ij} > Q_{ij}$ for each $1 \le i, j \le l$.

$\lim_{x\to x_0} Q(x) = Q$ means $\lim_{x\to x_0} Q_{ij}(x) = Q_{ij}$ for each $1 \le i, j \le l$.

The matrix P is called positive if all entries of P are non-negative and there exists at least one positive entry.

The matrix P is called zero if all entries of P are zero.

The matrix P is called cooperative (or quasi-positive) if all off-diagonal entries of P are non-negative, i.e., $P_{ij} \ge 0$ for $i \ne j$.

Moreover, $(d_1, \ldots, d_n) \to (0, \ldots, 0)$ means $\max_{1\le j=1\le n} d_j \to 0$.

$(d_1, \ldots, d_n) \to (\infty, \ldots, \infty)$ means $\min_{1\le j\le n} d_j \to \infty$.

The remaining part of the paper is organized as follows. In Section 2, we show some preliminaries for further applications. In Section 2 and 3, We show the asymptotic profiles of R_0 for model (1.3) as $(d_1, \cdots, d_n) \to (0, \ldots, 0)$ and $(d_1, \ldots, d_n) \to (\infty, \ldots, \infty)$, respectively. In Section 4, we apply the theoretical results to some concrete examples.

2 Some preliminaries

In this section, we recall the definition of basic reproduction number for reaction-diffusion epidemic models in [42]. Assume that the population $u = (u_1, \ldots, u_n)^T$ of model (1.3) is divided into two types: infected compartments, labeled by $i = 1, 2, \ldots, m$, and uninfected compartments, labeled by $i = m + 1, \ldots, n$. We set

$$\begin{aligned}
&u_I = (u_1, \ldots, u_m)^T, \;\; u_S = (u_{m+1}, \ldots, u_n)^T,\\
&d_I = (d_1, \ldots, d_m)^T, \;\; d_S = (d_{m+1}, \ldots, d_n)^T,\\
&d_I \Delta u_I = (d_1\Delta u_1, \ldots, d_m \Delta u_m)^T, \;\; d_S \Delta u_S = (d_{m+1}\Delta u_{m+1}, \ldots, d_n \Delta u_n)^T,\\
&f_I(x, u) = (f_1(x, u), \ldots, f_m(x, u))^T, \;\; f_S(x, u) = (f_{m+1}(x, u), \ldots, f_n(x, u))^T.
\end{aligned} \tag{2.1}$$

Let

$$U_s := \{u \ge 0 : u_i = 0 \text{ for any } i = 1, \ldots, m\}$$

denote the set of all disease-free states of (1.3), and assume that model (1.3) has a disease-free steady state

$$u^0(x) = \left(0, \ldots, 0, u^0_{m+1}(x), \ldots, u^0_n(x)\right)^T, \tag{2.2}$$

where $u^0_i(x) > 0$ for any $i = m+1, \ldots, n$ and $x \in \overline{\Omega}$. Define the following three matrices:

$$\begin{aligned} F(x,u) &= (F_{ij}(x,u))_{1\le i,j\le m} = \left(\frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\ V(x,u) &= (V_{ij}(x,u))_{1\le i,j\le m} = \left(\frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\ M(x,u) &= (M_{ij}(x,u))_{1\le i,j\le n-m} = \left(\frac{\partial f_{i+m}(x,u)}{\partial u_{j+m}}\right)_{1\le i,j\le n-m}, \end{aligned} \tag{2.3}$$

and let

$$B := d_I\Delta - V\left(x, u^0(x)\right). \tag{2.4}$$

The following assumptions are imposed on model (1.3): (see assumptions (A1)-(A6) in [42])

(A1) For each $1 \le i \le n$, functions $\mathcal{F}_i(x,u)$, $\mathcal{V}_i^+(x,u)$, $\mathcal{V}_i^-(x,u)$ are non-negative and continuously differentiable on $\overline{\Omega} \times \mathbb{R}^n_+$.

(A2) If $u_i = 0$, then $\mathcal{V}_i^- = 0$.

(A3) $\mathcal{F}_i = 0$ for $i > m$.

(A4) If $u \in U_s$, then $\mathcal{F}_i = \mathcal{V}_i^+ = 0$ for $i = 1, \ldots, m$.

(A5) $M(x, u^0(x))$ is cooperative for any $x \in \overline{\Omega}$, and

$$s\left(d_S\Delta + M(x, u^0(x))\right) < 0.$$

(A6) $-V(x, u^0(x))$ is cooperative for any $x \in \overline{\Omega}$, and $s\left(B\right) = s(d_I\Delta - V\left(x, u^0(x)\right)) < 0$.

Assumptions (A1)-(A6) are satisfied for most reaction-diffusion epidemic models.

Denote

$$X = C\left(\overline{\Omega}, \mathbb{R}^m\right) \quad \text{and} \quad X_+ = C\left(\overline{\Omega}, \mathbb{R}^m_+\right). \tag{2.5}$$

X is an ordered Banach space, and X_+ is a positive cone with nonempty interior. Let $T(t)$ be the semigroup generated by B on X, i.e., $T(t)$ is the solution semigroup associated with the following linear reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u_I}{\partial t} = d_I \Delta u_I - V\left(x, u^0(x)\right) u_I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.6}$$

It follows from the comparison principle (see [38, Theorem 3.12]) and assumption (A6) that B is resolvent-positive, $T(t)$ is positive (i.e., $T(t)X_+ \subset X_+$ for all $t > 0$), $s(B) < 0$, and $-B^{-1}\phi = \int_0^\infty T(t)\phi dt$ for $\phi \in X$. Note that $F\left(x, u^0(x)\right)$ is a positive matrix, and it can also be viewed as a positive operator on $C\left(\overline{\Omega}, \mathbb{R}^m\right)$:

$$\phi \in C\left(\overline{\Omega}, \mathbb{R}^m\right) \mapsto F\left(x, u^0(x)\right) \phi.$$

Clearly, the linear operator $B + F(x, u^0(x))$ is also resolvent-positive. Then it follows from [42, Section 3] (or [38, Theorem 3.5]) that:

Proposition 2.1. *Assume that (A1)-(A6) hold. Then the basic reproduction number is defined by*

$$R_0 = r\left(-F(x, u^0(x))B^{-1}\right).$$

Moreover, the following statements hold.

(i) $R_0 - 1$ *has the same sign as* $s\left(B + F(x, u^0(x))\right)$.

(ii) If $R_0 < 1$, *then* $u^0(x)$ *is locally asymptotically stable for system* (1.3).

Next we recall several results which will be used later. First we have the following the comparison principle.

Lemma 2.2. *Assume that* $P_i(x)$ $(i = 1, 2)$ *are* $m \times m$ *cooperative matrices for any* $x \in \overline{\Omega}$, *all entries of* $P_i(x)$ $(i = 1, 2)$ *are continuous, and* $P_1(x) \geq P_2(x)$. *Let* $T_i(t)$ *be the solution semigroup on* X *(defined in Eq.* (2.5)*) associated with the following linear reaction-diffusion system:*

$$\begin{cases} \dfrac{\partial u_I}{\partial t} = d_I \Delta u_I + P_i(x) u_I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.7}$$

where $d_I\Delta u_I$ is defined as in (2.1)*, and $d_i > 0$ for $i = 1, \dots, m$. Then $T_1(t)\phi \geq T_2(t)\phi$ for any $\phi \in X_+$ and $t > 0$.*

Proof. Denote $U_i(x,t) = T_i(t)\phi$ for $\phi \in X_+$, and it follows from the comparison principle of cooperative parabolic systems that $U_i(x,t) \geq 0$ for any $(x,t) \in \overline{\Omega} \times (0,\infty)$ and $i = 1, 2$. Let $W(x,t) = U_1(x,t) - U_2(x,t)$, and then $W(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t} = d_I \Delta W + P_2(x)W + (P_1(x) - P_2(x))\, U_1, & x \in \Omega,\ t > 0, \\ \dfrac{\partial W}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ W(x,0) = 0, & x \in \Omega. \end{cases} \tag{2.8}$$

Note that $P_1(x) \geq P_2(x)$ and $U_1(x,t) \geq 0$ for any $(x,t) \in \overline{\Omega} \times (0,\infty)$. Again it follows from the comparison principle of cooperative parabolic systems that $W(x,t) \geq 0$ for any $(x,t) \in \overline{\Omega} \times (0,\infty)$. This completes the proof. □

Secondly we recall the Krein-Rutmann theorem, (see [3, Theorems 3.1 and 3.2] or [31, Theorem 2.5]).

Lemma 2.3. (*i*) *Suppose that $T : X \to X$ is a positive compact linear operator with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue of T with an eigenvector in $X_+ \setminus \{0\}$.*

(*ii*) *Suppose that $T : X \to X$ is a strongly positive compact linear operator. Then $r(T)$ is positive and is a simple eigenvalue of T with an eigenvector in $Int(X_+)$, and there is no other eigenvalue with non-negative eigenvector. Moreover, if $S : X \to X$ is a linear operator such that $S - T$ is strongly positive, then $r(S) > r(T)$.*

Based on the Krein-Rutmann theorem in Lemma 2.3, we have the following two results.

Lemma 2.4. *Let L_1 and L_2 be bounded linear operators on X (defined in Eq.* (2.5)*). Assume that $L_1\phi \geq L_2\phi$ for any $\phi \in X_+$, and L_2 is a positive compact operator with positive spectral radius $r(L_2)$. Then $r(L_1) \geq r(L_2)$.*

Proof. It follows from Lemma 2.3 that $r(L_2)$ is an eigenvalue of L_2, and there exists $\phi \in X_+ \setminus \{0\}$ such that $\|\phi\|_\infty = 1$ and $L_2\phi = r(L_2)\phi$. Then $L_1^n\phi \geq r^n(L_2)\phi$, which implies that $\|L_1^n\| \geq r^n(L_2)$. Therefore, $r(L_1) = \lim_{n\to\infty} \|L_1^n\|^{1/n} \geq r(L_2)$. □

Consider the following eigenvalue problem:

$$\begin{cases} d_I\Delta\Phi - P(x)\Phi + aQ(x)\Phi = \lambda\Phi, & x \in \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.9}$$

where

$$\Phi = (\phi_1, \dots, \phi_m)^T, \quad d_I\Delta\Phi = (d_1\Delta\phi_1, \dots, d_m\Delta\phi_m)^T, \tag{2.10}$$

$a > 0$, $d_i > 0$ for $i = 1, \dots, m$, and $P(x) = (P_{ij}(x))_{1\leq i,j\leq m}$ and $Q(x) = (Q_{ij}(x))_{1\leq i,j\leq m}$ are $m \times m$ matrices with continuous entries. Recall that an eigenvalue λ of (2.9) is called the principal eigenvalue if $\lambda \in \mathbb{R}$ and for any eigenvalue such that $\tilde{\lambda} \neq \lambda$, we have $\mathcal{R}e\tilde{\lambda} < \lambda$.

Lemma 2.5. *Assume that $-P(x)$ is cooperative, $Q(x)$ is positive for any $x \in \overline{\Omega}$, and for any $a \in (0, \infty)$, there exists $x_a \in \Omega$ such that $-P(x_a) + aQ(x_a)$ is irreducible. Let $\lambda(a)$ be the principal eigenvalue of (2.9). Then $\lambda(a)$ is strictly increasing for $a \in (0, \infty)$.*

Proof. Since $-P(x) + aQ(x)$ is cooperative for any $x \in \overline{\Omega}$ and $a > 0$, it follows from Lemma 2.3 that $\lambda(a)$ is well defined and

$$\lambda(a) = \sup\{\mathcal{R}e\lambda : \lambda \text{ is an eigenvalue of problem (2.9)}\}.$$

Let $T^a(t)$ be the solution semigroup associated with the linear parabolic system

$$\begin{cases} \dfrac{\partial V}{\partial t} = d_I\Delta V - P(x)V + aQ(x)V, & t > 0,\ x \in \Omega, \\ \dfrac{\partial V}{\partial\nu} = 0, & t > 0,\ x \in \partial\Omega, \\ V(x,0) = V_0(x), & x \in \Omega. \end{cases} \tag{2.11}$$

Then it follows from [37, Theorem 7.4.1] that $T^a(t)$ is strongly positive and compact for any $a > 0$ and $t > 0$. Let $a_1 > a_2$, $\Phi \in X_+ \setminus \{0\}$, and

$$\begin{aligned} U_1(x,t) &= \left(U_1^{(1)}(x,t), \dots, U_1^{(m)}(x,t)\right)^T = T^{a_1}(t)\Phi, \\ U_2(x,t) &= \left(U_2^{(1)}(x,t), \dots, U_2^{(m)}(x,t)\right)^T = T^{a_2}(t)\Phi. \end{aligned} \tag{2.12}$$

Then $U_1(x,t), U_2(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t > 0$. It follows from Lemma 2.2 that $U_1(x,t) \geq U_2(x,t)$ for any $x \in \overline{\Omega}$ and $t > 0$. Let $W(x,t) = U_1(x,t) - U_2(x,t)$, and we see that $U(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t} = d_I \Delta W - P(x)W + a_2 Q(x)W + (a_1 - a_2)Q(x)U_1, & t > 0,\ x \in \Omega, \\ \dfrac{\partial W}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \\ W(x,0) = 0, & x \in \Omega. \end{cases} \quad (2.13)$$

Note that $Q(x)$ is positive for any $x \in \overline{\Omega}$, and $U_1(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t > 0$. Then there exist $1 \leq i_1 \leq n$ and $x_0 \in \Omega$ such that $\sum_{j=1}^{n} Q_{i_1 j}(x_0) U_1^{(j)}(x_0,t) > 0$ for any $t > 0$, and consequently $W_{i_1}(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t > 0$. Note that there exists $x_{a_2} \in \Omega$ such that $-P(x_{a_2}) + a_2 Q(x_{a_2})$ is irreducible. Then there exists $i_2 \neq i_1$ such that $-P_{i_2 i_1}(x_{a_2}) + a_2 Q_{i_2 i_1}(x_{a_2}) > 0$, which implies that $W_{i_2}(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t > 0$. Following the above process, we could obtain that $W(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t > 0$, which implies that $T_{a_1}(t) - T_{a_2}(t)$ is strongly positive for any $t > 0$. It follows from Lemma 2.3 that

$$r\left(T_{a_1}(t)\right) = e^{\lambda(a_1)t} > r\left(T_{a_2}(t)\right) = e^{\lambda(a_2)t} \text{ for any } t > 0,$$

which implies that $\lambda(a_1) > \lambda(a_2)$. This completes the proof. □

3 The effect of diffusion rates

In this section, we show the asymptotic profile of R_0 for model (1.3) when all the diffusion rates are large or small.

3.1 Small diffusion rates

In this subsection, we consider the asymptotic profile of R_0 when $(d_1, \ldots, d_n) \to (0, \ldots, 0)$. We first impose an additional assumption for this case:

(A7) The disease-free steady state $(0, \ldots, 0, u^0_{m+1}(x), \ldots, u^0_n(x))$ (defined in Eq. (2.2)) satisfies

$$\lim_{(d_{m+1},\ldots,d_n)\to(0,\ldots,0)} (u^0_{m+1}(x), \ldots, u^0_n(x)) = (c_{m+1}(x), \ldots, c_n(x)) \text{ in } C\left(\overline{\Omega}, \mathbb{R}^{n-m}\right), \tag{3.1}$$

where $c_k(x) > 0$ for any $x \in \overline{\Omega}$ and $k = m+1, \ldots, n$.

In the next section, we will show that this assumption is not restrictive, and it is satisfied for many kinds of epidemic models. Denote

$$c(x) = (0, \ldots, 0, c_{m+1}(x), \ldots, c_n(x)) \in C\left(\overline{\Omega}, \mathbb{R}^n\right), \tag{3.2}$$

and denote, for sufficiently small ϵ ($0 < \epsilon < \min\{c_i(x) : i = m+1, \ldots, n, x \in \overline{\Omega}\}$),

$$\begin{aligned}
\mathcal{D}^c_\epsilon &= \{(x, u_1, \ldots, u_n) : x \in \overline{\Omega},\ u_i = 0 \text{ for } i = 1, \ldots, m, \\
&\qquad u_i \in [c_i(x) - \epsilon, c_i(x) + \epsilon] \text{ for } i = m+1, \ldots, n\}, \\
\underline{V}^c_\epsilon &= \left(\min_{(x,u)\in\mathcal{D}^c_\epsilon} V_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\min_{(x,u)\in\mathcal{D}^c_\epsilon} \frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\
\overline{V}^c_\epsilon &= \left(\max_{(x,u)\in\mathcal{D}^c_\epsilon} V_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\max_{(x,u)\in\mathcal{D}^c_\epsilon} \frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\
\underline{F}^c_\epsilon &= \left(\min_{(x,u)\in\mathcal{D}^c_\epsilon} F_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\min_{(x,u)\in\mathcal{D}^c_\epsilon} \frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\
\overline{F}^c_\epsilon &= \left(\max_{(x,u)\in\mathcal{D}^c_\epsilon} F_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\max_{(x,u)\in\mathcal{D}^c_\epsilon} \frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}.
\end{aligned} \tag{3.3}$$

Since $\mathcal{D}^c_{\epsilon_1} \subset \mathcal{D}^c_{\epsilon_2}$ for $0 \le \epsilon_1 < \epsilon_2$, it follows that $\overline{F}^c_\epsilon$ and $\overline{V}^c_\epsilon$ are monotone decreasing for $\epsilon \ge 0$, and $\underline{F}^c_\epsilon$ and $\underline{V}^c_\epsilon$ are monotone increasing for $\epsilon \ge 0$. We will show that these functions $\overline{F}^c_\epsilon$, $\overline{V}^c_\epsilon$, $\underline{F}^c_\epsilon$ and $\underline{V}^c_\epsilon$ are also continuous for $\epsilon \ge 0$ in the Appendix.

Clearly, for $\epsilon = 0$, we have

$$\begin{aligned}
\underline{V}^c_0 &= \left(\min_{x\in\overline{\Omega}} V_{ij}(x, c(x))\right)_{1\le i,j\le m} = \left(\min_{x\in\overline{\Omega}} \frac{\partial \mathcal{V}_i(x, c(x))}{\partial u_j}\right)_{1\le i,j\le m}, \\
\overline{V}^c_0 &= \left(\max_{x\in\overline{\Omega}} V_{ij}(x, c(x))\right)_{1\le i,j\le m} = \left(\max_{x\in\overline{\Omega}} \frac{\partial \mathcal{V}_i(x, c(x))}{\partial u_j}\right)_{1\le i,j\le m}, \\
\underline{F}^c_0 &= \left(\min_{x\in\overline{\Omega}} F_{ij}(x, c(x))\right)_{1\le i,j\le m} = \left(\min_{x\in\overline{\Omega}} \frac{\partial \mathcal{F}_i(x, c(x))}{\partial u_j}\right)_{1\le i,j\le m}, \\
\overline{F}^c_0 &= \left(\max_{x\in\overline{\Omega}} F_{ij}(x, c(x))\right)_{1\le i,j\le m} = \left(\max_{x\in\overline{\Omega}} \frac{\partial \mathcal{F}_i(x, c(x))}{\partial u_j}\right)_{1\le i,j\le m}.
\end{aligned}$$

Now we show the asymptotic profile of R_0 as $(d_1, \ldots, d_n) \to (0, \ldots, 0)$, and the method is motivated by the one in [31].

Theorem 3.1. *Assume that (A1)-(A5) and (A7) hold,*

$$s\left(\overline{V}_0^c\right) < 0,\ s\left(-\underline{V}_0^c\right) < 0 \ \textit{and}\ r((\overline{V}_0^c)^{-1}\underline{F}_0^c) > 0,$$

and there exists $\epsilon_0 > 0$ such that, for any $x \in \overline{\Omega}$, $-\overline{V}_{\epsilon_0}^c$ is cooperative and $\underline{F}_{\epsilon_0}^c$ is positive, where $\underline{V}_\epsilon^c$, $\overline{V}_\epsilon^c$ and $\underline{F}_\epsilon^c$ are defined in (3.3)*. If the matrix $-V(x, c(x)) + aF(x, c(x))$ is irreducible for any $a > 0$ and $x \in \overline{\Omega}$, where $c(x)$ is defined in* (3.2)*, then*

$$\lim_{(d_1,\ldots,d_n)\to(0,\ldots,0)} R_0 = R_0^c := \max_{x\in\overline{\Omega}} \left[r\left(-V^{-1}(x, c(x))F(x, c(x))\right)\right].$$

Proof. **Step 1.** We show that there exist positive constants $\underline{R}_0^c$, $\overline{R}_0^c$ and C_2 such that $R_0 \in [\underline{R}_0^c, \overline{R}_0^c]$ for any $d_1, \ldots, d_m > 0$ and $d_{m+1}, \ldots, d_n \in (0, C_2)$.
Since $-\overline{V}_{\epsilon_0}^c$ is cooperative and $\underline{F}_{\epsilon_0}^c$ is positive for any $x \in \overline{\Omega}$, it follows from the monotonicity of $\overline{F}_\epsilon^c$, $\overline{V}_\epsilon^c$, $\underline{F}_\epsilon^c$ and $\underline{V}_\epsilon^c$ that $-\overline{V}_\epsilon^c$ and $-\underline{V}_\epsilon^c$ are cooperative, and $\overline{F}_{\epsilon_0}^c$ and $\underline{F}_\epsilon^c$ are positive for any $\epsilon \in [0, \epsilon_0]$. Note that $\underline{V}_\epsilon^c$ and $\overline{V}_\epsilon^c$ are continuous with respect to ϵ (see Proposition 5.1), and

$$\lim_{\epsilon\to 0} \underline{V}_\epsilon^c = \underline{V}_0^c \ \text{ and } \ \lim_{\epsilon\to 0} \overline{V}_\epsilon^c = \overline{V}_0^c.$$

It follows from [20, Theorem 2.5.1] that there exists $\epsilon_1 \in (0, \epsilon_0)$ such that

$$s(-\underline{V}_\epsilon^c) < 0,\ s(-\overline{V}_\epsilon^c) < 0 \ \text{ for any } \ \epsilon \in (0, \epsilon_1].$$

Similarly, $\left(\overline{V}_\epsilon^c\right)^{-1} \underline{F}_\epsilon^c$ is continuous with respect to ϵ for $\epsilon \in (0, \epsilon_1)$, and there exists $\epsilon_2 \in (0, \epsilon_1)$ such that $r\left(\left(\overline{V}_\epsilon^c\right)^{-1} \underline{F}_\epsilon^c\right) > 0$ for any $\epsilon \in (0, \epsilon_2]$. It follows from (A7) that, for the above given $\epsilon_2 > 0$, there exists $C_2 > 0$ such that

$$c_i(x) - \epsilon_2 \le u_i^0(x) \le c_i(x) + \epsilon_2$$

for any $x \in \overline{\Omega}$, $d_{m+1}, \ldots, d_n \in (0, C_2)$ and $i = m+1, \ldots, n$. Denote by $\overline{T}_{\epsilon_2}^c(t)$, $\underline{T}_{\epsilon_2}^c(t)$ and $T(t)$ the semigroups generated by $d_I\Delta - \overline{V}_{\epsilon_2}^c$, $d_I\Delta - \underline{V}_{\epsilon_2}^c$ and $d_I\Delta - V(x, u^0(x))$, respectively. Note that

$$-\overline{V}_{\epsilon_2}^c \le -V(x, u^0(x)) \le -\underline{V}_{\epsilon_2}^c \tag{3.4}$$

for any $d_1, \ldots, d_m > 0$ and $d_{m+1}, \ldots, d_n \in (0, C_2)$, and $-\overline{V}^c_{\epsilon_2}$ is cooperative for any $x \in \overline{\Omega}$. Then it follows from Lemma 2.2 that for any $\phi \in X_+$ (defined in Eq. (2.5)), $d_1, \ldots, d_m > 0$ and $d_{m+1}, \ldots, d_n \in (0, C_2)$,

$$\overline{T}^c_{\epsilon_2}(t)\phi \leq T(t)\phi \leq \underline{T}^c_{\epsilon_2}(t)\phi. \tag{3.5}$$

Note that

$$s\left(d_I\Delta - \underline{V}^c_{\epsilon_2}\right) = s(-\underline{V}^c_{\epsilon_2}) < 0, \;\; s\left(d_I\Delta - \overline{V}^c_{\epsilon_2}\right) = s\left(-\overline{V}^c_{\epsilon_2}\right) < 0.$$

This, combined with Lemma 2.3 and the spectral mapping theorem, implies that $r\left(T^c_{\epsilon_2}(t)\right), r\left(T^c_{\epsilon_2}(t)\right) \in (0,1)$. Therefore, for any $d_1, \ldots, d_m > 0$ and $d_{m+1}, \ldots, d_n \in (0, C_2)$, $s\left(d_I\Delta - V(x, u^0(x))\right) < 0$, which implies that assumption (A6) is satisfied for any $d_1, \ldots, d_m > 0$ and $d_{m+1}, \ldots, d_n \in (0, C_2)$. It follows from Eq. (3.5) that

$$\underline{F}^c_{\epsilon_2} \int_0^\infty \overline{T}^c_{\epsilon_2}(t)\phi dt \leq F(x, u^0(x)) \int_0^\infty T(t)\phi dt \leq \overline{F}^c_{\epsilon_2} \int_0^\infty \underline{T}^c_{\epsilon_2}(t)\phi dt.$$

It follows from [42, Theorem 3.4] that

$$r\left(\underline{F}^c_{\epsilon_2} \int_0^\infty \overline{T}^c_{\epsilon_2} dt(t)\right) = r\left(\left(\overline{V}^c_{\epsilon_2}\right)^{-1} \underline{F}^c_{\epsilon_2}\right) > 0,$$

and $\underline{F}^c_{\epsilon_2}$ is positive and not zero for any $x \in \overline{\Omega}$. Then we see from Lemma 2.4 that, for any $d_1, \ldots, d_m > 0$ and $d_{m+1}, \ldots, d_n \in (0, C_2)$,

$$r\left(\left(\overline{V}^c_{\epsilon_2}\right)^{-1} \underline{F}^c_{\epsilon_2}\right) \leq R_0 \leq r\left(\overline{F}^c_{\epsilon_2} \int_0^\infty \underline{T}^c_{\epsilon_2}(t)\phi dt\right) = r\left(\left(\underline{V}^c_{\epsilon_2}\right)^{-1} \overline{F}^c_{\epsilon_2}\right).$$

Let $\underline{R}^c_0 = r\left(\left(\overline{V}^c_{\epsilon_2}\right)^{-1} \underline{F}^c_{\epsilon_2}\right)$ and $\overline{R}^c_0 = r\left(\left(\underline{V}^c_{\epsilon_2}\right)^{-1} \overline{F}^c_{\epsilon_2}\right)$. This completes the proof for Step 1.

Step 2. For any $x \in \overline{\Omega}$, denote

$$\begin{aligned}
\mathcal{D}^x &= \{(u_1, \dots, u_n): \ u_i = 0 \text{ for } i = 1, \dots, m, \\
&\qquad u_i \in [c_i(x) - \epsilon, c_i(x) + \epsilon] \text{ for } i = m+1, \dots, n\}, \\
\underline{V}^x_\epsilon &= \left(\min_{u \in \mathcal{D}^x} V_{ij}(x,u)\right)_{1 \le i,j \le m} = \left(\min_{u \in \mathcal{D}^x} \frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1 \le i,j \le m}, \\
\overline{V}^x_\epsilon &= \left(\max_{u \in \mathcal{D}^x} V_{ij}(x,u)\right)_{1 \le i,j \le m} = \left(\max_{u \in \mathcal{D}^x} \frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1 \le i,j \le m}, \\
\underline{F}^x_\epsilon &= \left(\min_{u \in \mathcal{D}^x} F_{ij}(x,u)\right)_{1 \le i,j \le m} = \left(\min_{u \in \mathcal{D}^x} \frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1 \le i,j \le m}, \\
\overline{F}^x_\epsilon &= \left(\max_{u \in \mathcal{D}^x} F_{ij}(x,u)\right)_{1 \le i,j \le m} = \left(\max_{u \in \mathcal{D}^x} \frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1 \le i,j \le m}.
\end{aligned} \tag{3.6}$$

We show that, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
\tilde{R}_0 &:= r\left((d_I\Delta - \underline{V}^x_\epsilon)^{-1}\, \overline{F}^x_\epsilon\right) \to \tilde{R}^0_0 := \max_{x \in \overline{\Omega}} r\left((\underline{V}^x_\epsilon)^{-1}\, \overline{F}^x_\epsilon\right), \\
\check{R}_0 &:= r\left(\left(d_I\Delta - \overline{V}^x_\epsilon\right)^{-1} \underline{F}^x_\epsilon\right) \to \check{R}^0_0 := \max_{x \in \overline{\Omega}} r\left(\left(\overline{V}^x_\epsilon\right)^{-1} \underline{F}^x_\epsilon\right),
\end{aligned} \tag{3.7}$$

as $d_I = (d_1, \dots, d_m) \to (0, \dots, 0)$.

We can view matrices $-\underline{V}^x_\epsilon + a\overline{F}^x_\epsilon$ and $-\overline{V}^x_\epsilon + a\underline{F}^x_\epsilon$ as matrix-valued functions of (x, ϵ, a). Then $-\underline{V}^x_\epsilon + a\overline{F}^x_\epsilon$ and $-\overline{V}^x_\epsilon + a\underline{F}^x_\epsilon$ are continuous and consequently uniformly continuous on $\overline{\Omega} \times [0, \epsilon_2] \times [1/\overline{R}_0, 1/\underline{R}_0]$ (see Proposition 5.1). This implies that

$$\left.\begin{aligned}
\lim_{\epsilon \to 0}(-\underline{V}^x_\epsilon + a\overline{F}^x_\epsilon) &= -V(x, c(x)) + aF(x, c(x)) \\
\lim_{\epsilon \to 0}(-\overline{V}^x_\epsilon + a\underline{F}^x_\epsilon) &= -V(x, c(x)) + aF(x, c(x))
\end{aligned}\right\} \text{ uniformly for } (x, a) \in \overline{\Omega} \times [1/\overline{R}_0, 1/\underline{R}_0]. \tag{3.8}$$

Therefore, there exists $\epsilon_3 < \epsilon_2$ such that for any $\epsilon \in (0, \epsilon_3)$, matrices $-\underline{V}^x_\epsilon + a\overline{F}^x_\epsilon$ and $-\overline{V}^x_\epsilon + a\underline{F}^x_\epsilon$ are irreducible for any $a \in [1/\overline{R}_0, 1/\underline{R}_0]$ and $x \in \overline{\Omega}$. In this step, we always assume that $\epsilon \in (0, \epsilon_3]$. Clearly,

$$-\overline{V}^c_\epsilon \le -\underline{V}^x_\epsilon \le -\underline{V}^c_\epsilon. \tag{3.9}$$

Noticing that $s(-\underline{V}^c_\epsilon), s(-\overline{V}^c_\epsilon) < 0$ and $-\underline{V}^c_\epsilon$ is cooperative for any $x \in \overline{\Omega}$, we have $s\left(d_I\Delta - \underline{V}^x_\epsilon\right) < 0$.

Clearly, $\tilde{R}_0 \in [\underline{R}_0, \overline{R}_0]$ and $\tilde{R}_0 > 0$. Let $\tilde{\kappa} = 1/\tilde{R}_0$, and it follows from Lemma 2.3 that $\tilde{R}_0$ is an eigenvalue of $(d_I\Delta - \underline{V}_\epsilon^x)^{-1}\overline{F}_\epsilon^x$ with a non-negative eigenvector $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_m)$. Clearly, $\tilde{\kappa}$ can be viewed as a function of d_I (or respectively $(d_1, \ldots, d_m)$), and

$$d_I\Delta\tilde{\phi} - \underline{V}_\epsilon^x\tilde{\phi} + \tilde{\kappa}(d_1, \ldots, d_m)\overline{F}_\epsilon^x\tilde{\phi} = 0.$$

Let $\delta = \delta(d_1, \ldots, d_m, a)$ be the principal eigenvalue of the auxiliary eigenvalue problem

$$d_I\Delta\phi - \underline{V}_\epsilon^x\phi + a\overline{F}_\epsilon^x\phi = \delta\phi. \tag{3.10}$$

Note that $\underline{V}_\epsilon^x + \tilde{\kappa}(d_1, \ldots, d_m)\overline{F}_\epsilon^x$ is irreducible. Then $\tilde{\phi} > 0$, and

$$\delta(d_1, \ldots, d_m, \tilde{\kappa}(d_1, \ldots, d_m)) = 0.$$

It follows from [23, Theorem 1.4] that

$$\lim_{(d_1,\ldots,d_m)\to(0,\ldots,0)} \delta(d_1, \ldots, d_m, a) = \max_{x\in\overline{\Omega}} \hat{\delta}\left(-\underline{V}_\epsilon^x + a\overline{F}_\epsilon^x\right).$$

Here $\hat{\delta}(Q)$ represents the eigenvalue of matrix Q with greatest real part. Define

$$\delta(d_1, \ldots, d_m, a) := \max_{x\in\overline{\Omega}} \hat{\delta}\left(-\underline{V}_\epsilon^x + a\overline{F}_\epsilon^x\right)$$

for $(d_1, \ldots, d_m) = (0, \ldots, 0)$. Then, for each $a \in [1/\overline{R}_0, 1/\underline{R}_0]$, $\delta(d_1, \ldots, d_m, a)$ is a continuous function of $(d_1, \ldots, d_m)$ on $\text{Int}(\mathbb{R}_+^m) \cup \{(0, \ldots, 0)\}$. It follows from Lemma 2.5 that $\delta(d_1, \ldots, d_m, a)$ is strictly increasing in a for each $(d_1, \ldots, d_m) > (0, \ldots, 0)$. Similarly, we see from Lemma 2.5 that, for each $x \in \overline{\Omega}$, $\hat{\delta}\left(-\underline{V}_\epsilon^x + a\overline{F}_\epsilon^x\right)$ is also strictly increasing in a. This implies that $\delta(d_1, \ldots, d_m, a)$ is also strictly increasing in a for $(d_1, \ldots, d_m) = (0, \ldots, 0)$. Since for any $x \in \overline{\Omega}$,

$$\underline{V}_{\epsilon_2} \leq \underline{V}_\epsilon^x \leq \overline{V}_{\epsilon_2}, \;\; \underline{F}_{\epsilon_2} \leq \overline{F}_\epsilon^x \leq \overline{F}_{\epsilon_2},$$

it follows from Step 1 that

$$\underline{R}_0 \leq r\left((\underline{V}_\epsilon^x)^{-1}\overline{F}_\epsilon^x\right) \leq \overline{R}_0,$$

for any $x \in \overline{\Omega}$, and

$$\tilde{R}_0 = r\left((d_I\Delta - \underline{V}_\epsilon^x)^{-1}\overline{F}_\epsilon^x\right) \in [\underline{R}_0, \overline{R}_0]. \tag{3.11}$$

Noticing that, for each $x \in \overline{\Omega}$,

$$\hat{\delta}\left(-\underline{V}_\epsilon^x + \frac{1}{r\left((\underline{V}_\epsilon^x)^{-1}\overline{F}_\epsilon^x\right)}\overline{F}_\epsilon^x\right) = 0.$$

Then the monotonicity of $\hat{\delta}\left(-\underline{V}_\epsilon^x + a\overline{F}_\epsilon^x\right)$ in a implies that, for any $x \in \overline{\Omega}$,

$$\hat{\delta}\left(-\underline{V}_\epsilon^x + \frac{1}{\tilde{R}_0^0}\overline{F}_\epsilon^x\right) \leq 0,$$

where $\tilde{R}_0^0$ is defined as in Eq. (3.7), and the equality holds if and only if x achieves the maximum point of $r\left((\underline{V}_\epsilon^x)^{-1}\overline{F}_\epsilon^x\right)$. Therefore, the monotonicity of $\delta(0, \ldots, 0, a)$ implies that the unique zero of

$$\delta(0, \ldots, 0, a) = \max_{x \in \overline{\Omega}} \hat{\delta}\left(-\underline{V}_\epsilon^x + a\overline{F}_\epsilon^x\right) = 0$$

on $[1/\overline{R}_0, 1/\underline{R}_0]$ is $a = 1/\tilde{R}_0^0$.

Now we claim that the first equation of (3.7) holds. If it is not true, then

$$\kappa(d_1, \ldots, d_m) \not\to 1/\tilde{R}_0^0 \text{ as } (d_1, \ldots, d_n) \to (0, \ldots, 0).$$

Noticing that $\kappa(d_1, \ldots, d_m)$ is bounded from Eq. (3.11), we see that there exists a sequence $\left\{\left(d_1^{(j)}, \ldots, d_m^{(j)}\right)\right\}_{j=1}^{\infty}$ and $\kappa_0 \left(\neq 1/\tilde{R}_0^0\right) \in [1/\overline{R}_0, 1/\underline{R}_0]$ such that

$$\left(d_1^{(j)}, \ldots, d_m^{(j)}\right) \to (0, \ldots, 0), \quad \kappa_n := \kappa\left(d_1^{(j)}, \ldots, d_m^{(j)}\right) \to \kappa_0 \text{ as } j \to \infty.$$

Without loss of generality, we assume that $\kappa_0 < 1/\tilde{R}_0^0$. Then there exist $\tilde{\epsilon}$ and j_0 such that $\kappa_0 + \tilde{\epsilon} < 1/\tilde{R}_0^0$ and $\kappa_j < \kappa_0 + \tilde{\epsilon}$ for any $j > j_0$. Then, for any $j > j_0$,

$$0 = \delta\left(d_1^{(j)}, \ldots, d_m^{(j)}, \kappa_j\right) < \delta\left(d_1^{(j)}, \ldots, d_m^{(j)}, \kappa_0 + \tilde{\epsilon}\right),$$

which yields

$$0 \leq \lim_{j \to \infty} \delta\left(d_1^{(j)}, \ldots, d_m^{(j)}, \kappa_0 + \tilde{\epsilon}\right) = \delta(0, \ldots, 0, \kappa_0 + \tilde{\epsilon}) < 0.$$

This is a contradiction, and therefore, the first equation of (3.7) holds. Similarly, we can prove that the second equation of (3.7) holds.

Step 3. We show that

$$\lim_{(d_1,\ldots,d_n)\to(0,\ldots,0)} R_0 = \max_{x\in\overline{\Omega}} \left[r\left(-V^{-1}(x,c(x))F(x,c(x))\right)\right].$$

Clearly, $(\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x$ can be viewed as a matrix-valued function of (x,ϵ), where $(x,\epsilon)\in\overline{\Omega}\times[0,\epsilon_3]$, and $(\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x$ is continuous on $\overline{\Omega}\times[0,\epsilon_3]$ (see Proposition 5.1). It follows from [20, Section 2.5.7] that $r\left((\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x\right)$ is continuous on $\overline{\Omega}\times[0,\epsilon_3]$, and consequently, $r\left((\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x\right)$ is uniformly continuous on $\overline{\Omega}\times[0,\epsilon_3]$. This implies that

$$\lim_{\epsilon\to 0} r\left((\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x\right) = r\left(-V^{-1}(x,c(x))F(x,c(x))\right) \quad \text{in} \quad C(\overline{\Omega}).$$

Then

$$\lim_{\epsilon\to 0}\tilde{R}_0 = \lim_{\epsilon\to 0}\max_{x\in\overline{\Omega}} r\left[\left((\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x\right)\right] = R_0^c = \max_{x\in\overline{\Omega}}\left[r\left(-V^{-1}(x,c(x))F(x,c(x))\right)\right].$$

Similarly, we can prove that

$$\lim_{\epsilon\to 0}\check{R}_0 = R_0^c.$$

For any $\epsilon\in(0,\epsilon_3)$, there exists $\delta>0$ such that for any $d_{m+1},\ldots,d_n<\delta$,

$$u_i^0(x)\in[c_i(x)-\epsilon, c_i+\epsilon] \;\text{ for any }\; i=m+1,\ldots,n \;\text{ and }\; x\in\overline{\Omega}.$$

Then

$$\check{R}_0 = r\left(\left(d_I\Delta-\overline{V}_\epsilon^x\right)^{-1}\underline{F}_\epsilon^x\right) \le R_0 \le \tilde{R}_0 = r\left((d_I\Delta-\underline{V}_\epsilon^x)^{-1}\,\overline{F}_\epsilon^x\right)$$

for any $d_1,\ldots,d_m>0$ and $d_{m+1},\ldots,d_n<\delta$. Therefore,

$$\check{R}_0 \le \liminf_{(d_1,\ldots,d_n)\to(0,\ldots,0)} R_0 \le \limsup_{(d_1,\ldots,d_n)\to(0,\ldots,0)} R_0 \le \tilde{R}_0^0.$$

Taking $\epsilon\to 0$, we see that

$$\lim_{(d_1,\ldots,d_n)\to(0,\ldots,0)} R_0 = R_0^c.$$

This completes the proof. □

Remark 3.2. In Theorem 3.1, we assume that there exists $\epsilon_0>0$ such that, for any $x\in\overline{\Omega}$, $-\overline{V}_{\epsilon_0}^c$ is cooperative and $\underline{F}_{\epsilon_0}^c$ is positive. In Section 4, we will show that in some concrete examples, any off-diagonal entry in $\overline{V}_\epsilon^c$, $\overline{V}_\epsilon^c$ either equals to zero or is strictly positive, and any entry of $\overline{F}_\epsilon^c$ or $\underline{F}_\epsilon^c$ is strictly positive. In that case we only need to assume that $-\overline{V}_0^c$ is cooperative and $\underline{F}_0^c$ is positive to obtain results in Theorem 3.1.

3.2 Large diffusion rates

In this subsection, we consider the asymptotic profile of R_0 when $(d_1, \ldots, d_n) \to (\infty, \ldots, \infty)$. For this case, we impose an additional assumption:

(A8) The disease-free equilibrium $(0, \ldots, 0, u^0_{m+1}(x), \ldots, u^0_n(x))$ (defined in Eq. (2.2)) satisfies

$$\lim_{(d_{m+1},\ldots,d_n)\to(\infty,\ldots,\infty)} (u^0_{m+1}(x), \ldots, u^0_n(x)) = (\tilde{u}_{m+1}, \ldots, \tilde{u}_n) \text{ in } C\left(\overline{\Omega}, \mathbb{R}^{n-m}\right), \tag{3.12}$$

where $\tilde{u}_k$ is a positive constant for $k = m+1, \ldots, n$.

We will also show that this assumption is not restrictive and it is satisfied for many kinds of epidemic models in the next section. Denote

$$\tilde{u} = (0, \ldots, 0, \tilde{u}_{m+1}, \ldots, \tilde{u}_n) \in \mathbb{R}^n, \tag{3.13}$$

and denote, for given sufficiently small ϵ $(0 < \epsilon < \min\{\tilde{u}_i : i = m+1, \ldots, n\})$,

$$\begin{aligned}
\mathcal{D} =& \{(u_1, \ldots, u_n) : u_i = 0 \text{ for } i = 1, \ldots, m, \\
&u_i \in [\tilde{u}_i - \epsilon, \tilde{u}_i + \epsilon] \text{ for } i = m+1, \ldots, n\}, \\
\underline{V}_\epsilon =& \left(\min_{x\in\overline{\Omega}, u\in\mathcal{D}} V_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\min_{x\in\overline{\Omega}, u\in\mathcal{D}} \frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\
\overline{V}_\epsilon =& \left(\max_{x\in\overline{\Omega}, u\in\mathcal{D}} V_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\max_{x\in\overline{\Omega}, u\in\mathcal{D}} \frac{\partial \mathcal{V}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\
\underline{F}_\epsilon =& \left(\min_{x\in\overline{\Omega}, u\in\mathcal{D}} F_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\min_{x\in\overline{\Omega}, u\in\mathcal{D}} \frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}, \\
\overline{F}_\epsilon =& \left(\max_{x\in\overline{\Omega}, u\in\mathcal{D}} F_{ij}(x,u)\right)_{1\le i,j\le m} = \left(\max_{x\in\overline{\Omega}, u\in\mathcal{D}} \frac{\partial \mathcal{F}_i(x,u)}{\partial u_j}\right)_{1\le i,j\le m}.
\end{aligned} \tag{3.14}$$

Similar to subsection 3.1, we could also prove that $\overline{F}_\epsilon$ and $\overline{V}_\epsilon$ are monotone decreasing for $\epsilon \geq 0$, and $\underline{F}_\epsilon$ and $\underline{V}_\epsilon$ is monotone increasing for $\epsilon \geq 0$. Moreover, when $\epsilon = 0$,

$$\begin{aligned}
\underline{V}_0 &= \left(\min_{x\in\overline{\Omega}} V_{ij}(x,\tilde{u})\right)_{1\le i,j\le m} = \left(\min_{x\in\overline{\Omega}} \frac{\partial \mathcal{V}_i(x,\tilde{u})}{\partial u_j}\right)_{1\le i,j\le m},\\
\overline{V}_0 &= \left(\max_{x\in\overline{\Omega}} V_{ij}(x,\tilde{u})\right)_{1\le i,j\le m} = \left(\max_{x\in\overline{\Omega}} \frac{\partial \mathcal{V}_i(x,\tilde{u})}{\partial u_j}\right)_{1\le i,j\le m},\\
\underline{F}_0 &= \left(\min_{x\in\overline{\Omega}} F_{ij}(x,\tilde{u})\right)_{1\le i,j\le m} = \left(\min_{x\in\overline{\Omega}} \frac{\partial \mathcal{F}_i(x,\tilde{u})}{\partial u_j}\right)_{1\le i,j\le m},\\
\overline{F}_0 &= \left(\max_{x\in\overline{\Omega}} F_{ij}(x,\tilde{u})\right)_{1\le i,j\le m} = \left(\max_{x\in\overline{\Omega}} \frac{\partial \mathcal{F}_i(x,\tilde{u})}{\partial u_j}\right)_{1\le i,j\le m}.
\end{aligned}$$

Now we show the asymptotic profile of R_0 as $(d_1, \dots, d_n) \to (\infty, \dots, \infty)$.

Theorem 3.3. *Assume that (A1)-(A5) and (A8) hold,*

$$s\left(\overline{V}_0\right) < 0,\ s\left(-\underline{V}_0\right) < 0 \ \text{ and } \ r(\overline{V}_0^{-1}\underline{F}_0) > 0,$$

and there exists $\epsilon_0 > 0$ such that, for any $x \in \overline{\Omega}$, $-\overline{V}_{\epsilon_0}$ is cooperative and $\underline{F}_{\epsilon_0}$ is positive, where $\underline{V}_\epsilon$, $\overline{V}_\epsilon$ and $\underline{F}_\epsilon$ are defined in Eq. (3.14)*. Let*

$$\begin{aligned}
\check{V} &= \left(\int_\Omega V_{ij}(x,\tilde{u})dx\right)_{1\le i,j\le m} = \left(\int_\Omega \frac{\partial \mathcal{V}_i(x,\tilde{u})}{\partial u_j}dx\right)_{1\le i,j\le m},\\
\check{F} &= \left(\int_\Omega F_{ij}(x,\tilde{u})dx\right)_{1\le i,j\le m} = \left(\int_\Omega \frac{\partial \mathcal{F}_i(x,\tilde{u})}{\partial u_j}dx\right)_{1\le i,j\le m}.
\end{aligned} \tag{3.15}$$

If $r(\check{V}^{-1}\check{F})$ is the unique eigenvalue of $\check{V}^{-1}\check{F}$ with an eigenvector in $\mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, then

$$\lim_{(d_1,\dots,d_n)\to(\infty,\dots,\infty)} R_0 = r\left(\check{V}^{-1}\check{F}\right).$$

Proof. As in the Step 1 of Theorem 3.1, we could prove that there exist positive constants $\underline{R}_0$, $\overline{R}_0$ and C_2 such that $R_0 \in [\underline{R}_0, \overline{R}_0]$ for any $d_1, \dots, d_m > 0$ and $d_{m+1}, \dots, d_n > C_2$. Let $\kappa = 1/R_0$, and κ can be viewed as function of $(d_1, \dots, d_n)$. Since $k(d_1, \dots, d_n)$ is bounded for any $d_1, \dots, d_m > 0$ and $d_{m+1}, \dots, d_n > C_2$. Then, for any sequence $\{(d_1^{(j)}, \dots, d_n^{(j)})\}_{j=1}^\infty$ satisfying $(d_1^{(j)}, \dots, d_n^{(j)}) \to (\infty, \dots, \infty)$ as$j \to \infty$, there exists a subsequence $\{(d_1^{(j_k)}, \dots, d_n^{(j_k)})\}_{j=1}^\infty$ such that $\lim_{k\to\infty} \kappa\left(d_1^{(j_k)}, \dots, d_n^{(j_k)}\right)$ exists and is positive, which is denoted by κ^*. For convenience, we denote $d_i^{(j_k)}$ by $d_i^{(k)}$ for each $k \geq 1$

and $i = 1, \ldots, n$. Without loss of generality, we assume that $d_i^{(k)} \geq C_2$ for any $k \geq 1$ and $i = m+1, \ldots, n$.

Let $\hat{\phi}^{(k)} = (\hat{\phi}_1^{(k)}, \ldots, \hat{\phi}_m^{(k)})^T \geq (0, \ldots, 0)^T$ be the corresponding eigenvector of operator

$$-\left(d_I \Delta - V(x, u^0(x))\right)^{-1} F(x, u^0(x))$$

with respect to eigenvalue $R_0(d_1^{(k)}, \ldots, d_n^{(k)})$, where $\|\hat{\phi}^{(k)}\|_\infty = 1$ for each $k \geq 1$. That is, for $i = 1, \ldots, m$,

$$\Delta \hat{\phi}_i^{(k)} + \frac{1}{d_i^{(k)}} \left[-\sum_{i=1}^{m} V_{ij}(x, u^0(x)) \hat{\phi}_j^{(k)} + \kappa \left(d_1^{(k)}, \ldots, d_n^{(k)} \right) \sum_{j=1}^{m} F_{ij}(x, u^0(x)) \hat{\phi}_j^{(k)} \right] = 0,$$

where $u^0(x)$ depends on $\left(d_{m+1}^{(k)}, \ldots, d_n^{(k)}\right)$. Then it follows from the L^p theory that there exists a subsequence $\{k_l\}_{l=1}^\infty$ such that $\lim_{l \to \infty} \hat{\phi}_i^{(k_l)} = c_i^\infty$ in $C(\overline{\Omega}, \mathbb{R})$ for each $i = 1, \ldots, m$, where c_i^∞ is a nonnegative constant, and $c^\infty := (c_1^\infty, \ldots, c_m^\infty)^T$ satisfies

$$|c^\infty| = 1, \quad \text{and} \quad \check{V} c^\infty = \kappa^* \check{F} c^\infty.$$

Then $1/\kappa^* = r(\check{V}^{-1}\check{F})$. This completes the proof. □

Remark 3.4. We remark that there always exists a decomposition

$$\{(\mathcal{F}_i(x, u), \mathcal{V}_i(x, u))\}_{i=1}^n$$

of $\{f_i(x, u)\}_{i=1}^n$ such that

$$f_i(x, u) = \mathcal{F}_i(x, u) - \mathcal{V}_i(x, u) \ \text{ for } \ i = 1, \ldots, n, \ \text{ and } \ rank(F) = 1.$$

Consequently $r(\check{V}^{-1}\check{F})$ is the unique eigenvalue of $\check{V}^{-1}\check{F}$ with an eigenvector in $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$. Moreover, different decompositions of $\{f_i(x, u)\}_{n=1}^\infty$ will not change the portion of parameter space that the disease vanishes or spreads. Actually, if there exist two decompositions

$$\left\{ \left(\mathcal{F}_i^{(j)}(x, u), \mathcal{V}_i^{(j)}(x, u) \right) \right\}_{i=1}^n \ (j = 1, 2),$$

then there exist two basic reproduction numbers $R_0^{(1)}$ and $R_0^{(2)}$. It follows from [38, Theorem 3.5] that $R_0^{(1)} - 1$ and $R_0^{(2)} - 1$ have the same signs.

Remark 3.5. In Theorem 3.3, we assume that there exists $\epsilon_0 > 0$ such that, for any $x \in \overline{\Omega}$, $-\overline{V}_{\epsilon_0}$ is cooperative and $\underline{F}_{\epsilon_0}$ is positive. In Section 4, we will show that in some concrete examples, any off-diagonal entry in $\overline{V}_\epsilon$, $\overline{V}_\epsilon$ equals to zero or is strictly positive, and any entry of $\overline{F}_\epsilon$ or $\underline{F}_\epsilon$ is strictly positive. Therefore we only need to show that $-\overline{V}_0$ is cooperative and $\underline{F}_0$ is positive to obtain results in Theorem 3.3.

4 Applications

In this section, we give some examples to show that the general results in Theorems 3.1 and 3.3 can be applied to many different reaction-diffusion epidemic models.

4.1 Vector-host epidemic models

We consider two vector-host epidemic models. The first is given by [15] to model the outbreak of Zika in Rio De Janerio:

$$\begin{cases} \dfrac{\partial H_i}{\partial t} - \delta_1 \Delta H_i = -\lambda(x) H_i + \sigma_1(x) H_u(x) V_i, & x \in \Omega,\ t > 0, \\ \dfrac{\partial V_i}{\partial t} - \delta_2 \Delta V_i = \sigma_2(x) V_u H_i - \mu(x)(V_u + V_i) V_i, & x \in \Omega,\ t > 0, \\ \dfrac{\partial V_u}{\partial t} - \delta_3 \Delta V_u = -\sigma_2(x) V_u H_i + \beta(x)(V_u + V_i) - \mu(x)(V_u + V_i) V_u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial H_i}{\partial \nu} = \dfrac{\partial V_i}{\partial \nu} = \dfrac{\partial V_u}{\partial \nu}, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{4.1}$$

where $H_u(x)$, $H_i(x,t)$, $V_i(x,t)$ and $V_u(x,t)$ are the densities of uninfected hosts, infected hosts, infected vectors and uninfected vectors at space x and time t, respectively, Ω is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\delta_1, \delta_2, \delta_3$ are positive constants, and $\lambda(x)$, $H_u(x)$, $\sigma_i(x)$ $(i = 1, 2)$, $\beta(x)$ and $\mu(x)$ are strictly positive and belong to $C^\alpha(\overline{\Omega})$. The asymptotic properties of R_0 for this model has been investigated in [31], see also [30] for the global dynamics. We revisit it to show that the main results in Section 3 can be applied to this model to determine the asymptotic behavior of basic reproduction number R_0.

Letting

$$n = 3,\ m = 2 \ \text{ and } \ (u_1, u_2, u_3) = (H_i, V_i, V_u),$$

we could use the framework in Section 3. It follows from [31] that model (4.1) has a unique disease-free steady state $u^0(x) = \left(0, 0, \hat{V}(x)\right)$, where $\hat{V}(x)$ satisfies

$$\lim_{\delta_3 \to 0} \hat{V}(x) = \frac{\beta(x)}{\mu(x)} \quad \text{and} \quad \lim_{\delta_3 \to \infty} \hat{V}(x) = \frac{\int_\Omega \beta(x)dx}{\int_\Omega \mu(x)dx} \quad \text{in} \ \ C(\overline{\Omega}). \tag{4.2}$$

This implies that assumptions (A7) and (A8) are satisfied. For model (4.1),

$$V(x,u) = \begin{pmatrix} \lambda(x) & -\sigma_1(x)H_u(x) \\ 0 & \mu(x)u_3 \end{pmatrix}, \quad F(x,u) = \begin{pmatrix} 0 & 0 \\ \sigma_2(x)u_3 & 0 \end{pmatrix}, \tag{4.3}$$

where $u = (u_1, u_2, u_3)^T$, and

$$B = (\delta_1\Delta, \delta_2\Delta)^T - V(x, u^0(x)). \tag{4.4}$$

Then the basic reproduction number is given by

$$R_0 = r\left(-F(x, u^0(x))B^{-1}\right). \tag{4.5}$$

Moreover, for model (4.1),

$$\tilde{u} = \left(0, 0, \frac{\int_\Omega \beta dx}{\int_\Omega \mu dx}\right) \ \text{and} \ \ c(x) = \left(0, 0, \frac{\beta(x)}{\mu(x)}\right),$$

and a direct computation implies that all the assumptions of Theorem 3.1 and 3.3 are satisfied. Then we have the following results.

Proposition 4.1. *For model* (4.1)*, the following statements hold.*

(i)

$$\lim_{(\delta_1,\delta_2,\delta_3)\to(\infty,\infty,\infty)} R_0 = \frac{\int_\Omega \sigma_1 H_u dx \int_\Omega \sigma_2 dx}{\int_\Omega \lambda dx \int_\Omega \mu dx}.$$

(ii)

$$\lim_{(\delta_1,\delta_2,\delta_3)\to(0,0,0)} R_0 = \max_{x\in\overline{\Omega}} \frac{\sigma_1(x)\sigma_2(x)H_u(x)}{\lambda(x)\mu(x)}.$$

Next we consider another vector-host epidemic model:

$$
\begin{cases}
\dfrac{\partial I}{\partial t} = d_1 \Delta I + \beta_s(x) S V - (b(x) + \gamma(x))\, I, & x \in \Omega,\ t > 0, \\
\dfrac{\partial V}{\partial t} = d_2 \Delta V + \beta_m(x) M I - c(x) V, & x \in \Omega,\ t > 0, \\
\dfrac{\partial S}{\partial t} = d_3 \Delta S + \lambda_1(x) - b(x) S + \gamma(x) I - \beta_s(x) S V, & x \in \Omega,\ t > 0, \\
\dfrac{\partial M}{\partial t} = d_4 \Delta M + \lambda_2(x) - c(x) M - \beta_m(x) M I, & x \in \Omega,\ t > 0, \\
\dfrac{\partial I}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = \dfrac{\partial S}{\partial \nu} = \dfrac{\partial M}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,
\end{cases} \tag{4.6}
$$

where $I(x,t)$, $V(x,t)$, $S(x,t)$ and $M(x,t)$ are the densities of infected hosts, infected vectors, susceptible hosts and susceptible vectors at space x and time t, respectively, Ω is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, d_1, d_2, d_3, d_4 are positive constants, and $\lambda_i(x)$ $(i = 1, 2)$, $\beta_s(x)$, $\beta_m(x)$, $b(x)$, $\gamma(x)$ and $c(x)$ are strictly positive and belong to $C^\alpha(\overline{\Omega})$. The model was originally an ODE model (i.e., $d_1 = d_2 = d_3 = d_4 = 0$) proposed by Feng and Velasco-Hernández [14], and R_0 of the ODE model was obtained in [14, 40].

Letting

$$
n = 4,\ m = 2 \ \text{ and } \ (u_1, u_2, u_3, u_4) = (I, V, S, M),
$$

we could use the framework in Section 3. The model (4.6) has a unique disease-free steady state

$$
u^0(x) = (0, 0, \hat{S}(x), \hat{M}(x)),
$$

where $\left(\hat{S}(x), \hat{M}(x)\right)$ satisfies

$$
\left.
\begin{aligned}
\lim_{(d_3,d_4)\to(0,0)} \left(\hat{S}(x), \hat{M}(x)\right) &= \left(\frac{\lambda_1(x)}{b(x)}, \frac{\lambda_2(x)}{c(x)}\right) \\
\lim_{(d_3,d_4)\to(0,0)} \left(\hat{S}(x), \hat{M}(x)\right) &= \left(\frac{\int_\Omega \lambda_1 dx}{\int_\Omega b dx}, \frac{\int_\Omega \lambda_2 dx}{\int_\Omega c dx}\right)
\end{aligned}
\right\} \text{ in } C(\overline{\Omega}, \mathbb{R}^2). \tag{4.7}
$$

This implies that assumptions (A7) and (A8) are satisfied. A direct computation implies that, for model (4.6),

$$
F(x,u) = \begin{pmatrix} 0 & \beta_s(x) u_3 \\ \beta_m(x) u_4 & 0 \end{pmatrix}, \quad V(x,u) = \begin{pmatrix} b(x) + \gamma(x) & 0 \\ 0 & c(x) \end{pmatrix} \tag{4.8}
$$

for $u=(u_1,u_2,u_3,u_4)^T$, and

$$B=(d_1\Delta, d_2\Delta)^T - V(x,u^0(x)).$$

Then the basic reproduction number is also given by (4.5). Finally for model (4.6),

$$\tilde{u}=\left(0,0,\frac{\int_\Omega \lambda_1 dx}{\int_\Omega bdx},\frac{\int_\Omega \lambda_2 dx}{\int_\Omega cdx}\right) \text{ and } c(x)=\left(0,0,\frac{\lambda_1(x)}{b(x)},\frac{\lambda_2(x)}{c(x)}\right).$$

It is easy to check that all the assumptions of Theorem 3.1 and 3.3 are satisfied. Then we have the following results.

Proposition 4.2. *For model* (4.6)*, the following statements hold.*

(i)

$$\lim_{(d_1,d_2,d_3,d_4)\to(\infty,\infty,\infty,\infty)} R_0=\sqrt{\frac{\int_\Omega \lambda_1 dx \int_\Omega \lambda_2 dx \int_\Omega \beta_s dx \int_\Omega \beta_m dx}{\int_\Omega bdx \left(\int_\Omega cdx\right)^2 \int_\Omega (b+\gamma)dx}}.$$

(ii)

$$\lim_{(d_1,d_2,d_3,d_4)\to(0,0,0,0)} R_0=\max_{x\in\overline{\Omega}}\sqrt{\frac{\lambda_1(x)\lambda_2(x)\beta_s(x)\beta_m(x)}{b(x)c^2(x)(b(x)+\gamma(x))}}.$$

4.2 Staged progression model

In this subsection, we consider a staged progression model proposed in [19]. This model has a single uninfected compartment, and the infected individuals could pass through several stages of the disease with changing infectivity. It could be applied to model the transmission of many disease, such as HIV/AIDS, see [19]. The original model was an ODE model, and the reproduction number was obtained in [17, 40]. Here we consider the associated reaction-diffusion case:

$$\begin{cases}\dfrac{\partial I_1}{\partial t}=d_1\Delta I_1+h(N)\left(\displaystyle\sum_{k=1}^m \beta_k(x)SI_k\right)-(\nu_1(x)+\gamma_1(x))I_1, & x\in\Omega,\ t>0,\\ \dfrac{\partial I_i}{\partial t}=d_i\Delta I_i+\nu_{i-1}(x)I_{i-1}-(\nu_i(x)+\gamma_i(x))I_i, & x\in\Omega,\ t>0,\ 2\le i\le m,\\ \dfrac{\partial I_{m+1}}{\partial t}=d_{m+2}\Delta I_{m+1}+\nu_m(x)I_m-\gamma_{m+1}I_{m+1}, & x\in\Omega,\ t>0,\\ \dfrac{\partial S}{\partial t}=d_{m+1}\Delta S+\lambda(x)-b(x)S-h(N)\left(\displaystyle\sum_{k=1}^m \beta_k(x)SI_k\right), & x\in\Omega,\ t>0,\\ \dfrac{\partial S}{\partial \nu}=\dfrac{\partial I_i}{\partial \nu}=0, & x\in\partial\Omega,\ t>0,\ 1\le i\le m,\end{cases}$$

where $N = S + \sum_{i=1}^{m} I_i$, $h(N) = N^{-\alpha}$ with $\alpha \in [0,1]$, $S(x,t)$ is the density of the susceptible individuals, $I_i (i = 1, \ldots, m+1)$ is the density of the infected individuals at stage i, Ω is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, d_i $(i = 1, \ldots, m+2)$ are positive constants, and $\lambda(x)$, $b(x)$, $\beta_i(x)$ $(i = 1, \ldots, m)$, $\nu_i(x)$ $(i = 1, \ldots, m)$, $\gamma_i(x)$ $(i = 1, \ldots, m+1)$ are strictly positive and belong to $C^{\alpha}(\overline{\Omega})$. Note that I_{m+1} decouples from the others, and consequently we could consider the following model

$$\begin{cases} \dfrac{\partial I_1}{\partial t} = d_1 \Delta I_1 + h(N)\left(\sum_{k=1}^{m} \beta_k(x) S I_k\right) - (\nu_1(x) + \gamma_1(x)) I_1, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I_i}{\partial t} = d_i \Delta I_i + \nu_{i-1}(x) I_{i-1} - (\nu_i(x) + \gamma_i(x)) I_i, & x \in \Omega,\ t > 0,\ 2 \le i \le m, \\ \dfrac{\partial S}{\partial t} = d_{m+1} \Delta S + \lambda(x) - b(x) S - h(N)\left(\sum_{k=1}^{m} \beta_k(x) S I_k\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I_i}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\ 1 \le i \le m. \end{cases} \tag{4.9}$$

Letting

$$n = m+1,\ (u_1, \ldots, u_m) = (I_1, \ldots, I_m) \ \text{ and } \ u_{m+1} = S,$$

we could use the framework in Section 3. The model (4.9) has a unique disease-free steady state

$$u^0(x) = (0, \ldots, 0, \hat{S}(x)),$$

where $\hat{S}(x)$ satisfies

$$\lim_{d_{m+1} \to 0} \hat{S}(x) = \frac{\lambda(x)}{b(x)} \ \text{ and } \ \lim_{d_{m+1} \to \infty} \hat{S}(x) = \frac{\int_\Omega \lambda dx}{\int_\Omega b dx} \ \text{ in } \ C(\overline{\Omega}, \mathbb{R}). \tag{4.10}$$

This implies that assumptions (A7) and (A8) are satisfied. For model (4.9),

$$V(x,u) = (V_{ij}(u))_{1 \le i,j \le m} \ \text{ and } \ F(x,u) = (F_{ij}(x,u))_{1 \le i,j \le m},$$

where for $u = (u_1, \ldots, u_m)^T$,

$$F_{ij}(x,u) = \begin{cases} h\left(\sum_{k=1}^{m+1} u_k\right) \beta_j(x) u_{m+1} + h'\left(\sum_{k=1}^{m+1} u_k\right)\left(\sum_{k=1}^{m} \beta_k(x) u_k\right) u_{m+1} & i = 1, 1 \le j \le m, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_{ij}(x,u)=\begin{cases}\nu_i(x)+\gamma_i(x) & 1\le i\le m, j=i,\\ -\nu_{i-1}(x) & 2\le i\le m, j=i-1,\\ 0 & \text{otherwise},\end{cases}$$

and

$$B=(d_1\Delta,\dots,d_m\Delta)^T-V(x,u^0(x)).$$

And the basic reproduction number is given by (4.5). Also for model (4.9),

$$\tilde{u}=\left(0,\dots,0,\frac{\int_\Omega \lambda dx}{\int_\Omega b dx}\right) \text{ and } c(x)=\left(0,\dots,0,\frac{\lambda(x)}{b(x)}\right).$$

Since this model is more complex, we show that all the assumptions of Theorems 3.1 and 3.3 are satisfied.

Lemma 4.3. *The following statements hold.*

(i) For any $a>0$ and $x\in\overline{\Omega}$, $-V(x,c(x))+aF(x,c(x))$ is irreducible.

(ii) $r(\check{V}^{-1}\check{F})$ is the unique positive eigenvalue of $\check{V}^{-1}\check{F}$, where $\check{V}^{-1}$ and $\check{F}$ are defined as in (3.15).

Proof. Let $Q_{ij}(x)=-V_{ij}(x,c(x))+aF_{ij}(x,c(x))$. Then a direct computation implies that

$$Q_{ij}(x)=\begin{cases} a\beta_1(x)\frac{\lambda(x)}{b(x)}h\left(\frac{\lambda(x)}{b(x)}\right)-\nu_1(x)-\gamma_1(x) & i=1, j=1,\\ a\beta_j(x)\frac{\lambda(x)}{b(x)}h\left(\frac{\lambda(x)}{b(x)}\right) & i=1,\ 2\le j\le m,\\ -\nu_i(x)-\gamma_i(x) & 2\le i\le m, j=i,\\ \nu_{i-1}(x) & 2\le i\le m, j=i-1,\\ 0 & \textit{otherwise}.\end{cases} \tag{4.11}$$

For $i=1$, $2\le j\le m$,

$$Q_{1j}=a\beta_j(x)\frac{\lambda(x)}{b(x)}h\left(\frac{\lambda(x)}{b(x)}\right)\neq 0,$$

and for any $2\le i\le m$, $j>i$,

$$Q_{i(i-1)}\cdots Q_{21}Q_{1j}=a\nu_{i-1}(x)\cdots\nu_1(x)\beta_j(x)\frac{\lambda(x)}{b(x)}h\left(\frac{\lambda(x)}{b(x)}\right)\neq 0.$$

Similarly, for $1 \leq j \leq m$, $i > j$,

$$Q_{i(i-1)}Q_{(i-1)(i-2)}\cdots Q_{(j+1)j} = \nu_{i-1}(x)\cdots\nu_j(x) \neq 0.$$

Therefore, $-V(x, c(x)) + aF(x, c(x))$ is irreducible for any $a > 0$ and $x \in \overline{\Omega}$. This completes the proof of part (i).

Let $\check{V}^{-1} = (\alpha_{ij})_{1\leq i,j\leq m}$. From [40] and a direct computation, we see that

$$\alpha_{ij} = \begin{cases} 0 & 1 \leq i \leq m,\ j > i, \\ \dfrac{1}{\int_\Omega(\nu_i + \gamma_i)dx} & 1 \leq i \leq m,\ j = i, \\ \dfrac{\prod_{k=j}^{i-1}\int_\Omega \nu_k dx}{\prod_{k=j}^{i}\int_\Omega(\nu_k + \gamma_k)dx} & 1 \leq i \leq m,\ j < i. \end{cases} \tag{4.12}$$

Let $\check{F}\check{V}^{-1} = (\tilde{\alpha}_{ij})_{1\leq i,j\leq m}$. Then $\tilde{a}_{ij} = 0$ for any $2 \leq i \leq, m$ and $1 \leq j \leq m$, and

$$\tilde{a}_{11} = \left(\sum_{j=1}^{m} \frac{\int_\Omega \beta_j dx \prod_{k=1}^{j-1}\int_\Omega \nu_k dx}{\prod_{k=1}^{j}\int_\Omega(\nu_k + \gamma_k)dx}\right) \frac{\int_\Omega \lambda dx}{\int_\Omega b dx} h\left(\frac{\int_\Omega \lambda dx}{\int_\Omega b dx}\right). \tag{4.13}$$

Therefore, $r(\check{V}^{-1}\check{F}) = \tilde{a}_{11}$ is the unique positive eigenvalue of $\check{V}^{-1}\check{F}$. This completes the proof of part (ii). □

The other assumptions of Theorems 3.1 and 3.3 are easy to verified, and we omit the proof. Then we have the following results.

Proposition 4.4. *Let R_0 be the basic reproduction number of model* (4.9)*. Then*

(i)

$$\lim_{(d_1,\ldots,d_{m+1})\to(0,\ldots,0)} R_0 = \max_{x\in\overline{\Omega}} \left(\sum_{j=1}^{m} \frac{\beta_j(x)\prod_{k=1}^{j-1}\nu_k(x)}{\prod_{k=1}^{j}(\nu_k(x) + \gamma_k(x))}\right) \frac{\lambda(x)}{b(x)} h\left(\frac{\lambda(x)}{b(x)}\right).$$

(ii)

$$\lim_{(d_1,\ldots,d_{m+1})\to(\infty,\ldots,\infty)} R_0 = \left(\sum_{j=1}^{m} \frac{\int_\Omega \beta_j dx \prod_{k=1}^{j-1}\int_\Omega \nu_k dx}{\prod_{k=1}^{j}\int_\Omega(\nu_k + \gamma_k)dx}\right) \frac{\int_\Omega \lambda dx}{\int_\Omega b dx} h\left(\frac{\int_\Omega \lambda dx}{\int_\Omega b dx}\right).$$

5 Appendix

In this part, we prove a result that verifies the continuity of functions $\underline{F}^c_\epsilon$, $\overline{F}^c_\epsilon$, $\underline{V}^c_\epsilon$, $\overline{V}^c_\epsilon$, $\underline{F}^x_\epsilon$, $\overline{F}^x_\epsilon$, $\underline{V}^x_\epsilon$ and $\overline{V}^x_\epsilon$, which are defined in Eqs. (3.3) and (3.6).

Proposition 5.1. *Let $f(x,u) \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ and $c(x) \in C(\overline{\Omega},\mathbb{R})$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$. Denote*

$$\mathcal{D}^c_\epsilon = \{(x,u) : x \in \overline{\Omega}, u \in [c(x)-\epsilon, c(x)+\epsilon]\},\ \ \mathcal{D}^x_\epsilon = \{u : u \in [c(x)-\epsilon, c(x)+\epsilon]\},$$

and

$$H(\epsilon) = \max_{(x,u)\in\mathcal{D}^c_\epsilon} f(x,u),\ \ G(x,\epsilon) = \max_{u\in\mathcal{D}^x_\epsilon} f(x,u).$$

Then $H(\epsilon) \in C([0,1],\mathbb{R})$ and $G(x,\epsilon) \in C(\overline{\Omega}\times[0,1],\mathbb{R})$.

Proof. We first consider the continuity of $H(\epsilon)$. Let

$$C_1 := \min_{x\in\overline{\Omega}} c(x) - 2 \ \text{ and } \ C_2 =: \max_{x\in\overline{\Omega}} c(x) + 2.$$

The continuity of $f(x,u)$ implies that $f(x,u)$ is uniformly continuous on $\overline{\Omega}\times[C_1,C_2]$. Then, for any give $\gamma > 0$, there exists $\delta > 0$ such that, for any $(x_1,u_1),(x_2,u_2) \in \overline{\Omega}\times[C_1,C_2]$ satisfying $|x_1 - x_2| < \delta$ and $|u_1 - u_2| < \delta$,

$$|f(x_1,u_1) - f(x_2,u_2)| < \gamma. \tag{5.1}$$

Assume that $0 \leq \epsilon_1 < \epsilon_2 \leq 1$ and $\epsilon_2 - \epsilon_1 < \delta$. Clearly, $H(\epsilon_1) \leq H(\epsilon_2)$. Noticing that $\mathcal{D}^c_{\epsilon_2}$ is compact, we see that there exists $(x_0,u_0) \in \mathcal{D}^c_{\epsilon_2}$ such that $H(\epsilon_2) = f(x_0,u_0)$. Then there exists (x_0,u_1) such that $(x_0,u_1) \in \mathcal{D}^c_{\epsilon_1}$ and $|u_1 - u_0| < \delta$. It follows from Eq. (5.1) that $f(x_0,u_0) < f(x_0,u_1) + \gamma$, which implies that $H(\epsilon_2) < H(\epsilon_1) + \gamma$. Then exchanging the position of ϵ_1 and ϵ_2, we can also obtain that, for any $0 \leq \epsilon_2 < \epsilon_1 \leq 1$ and $\epsilon_1 - \epsilon_2 < \delta$,

$$H(\epsilon_2) \leq H(\epsilon_1) \leq H(\epsilon_2) + \gamma.$$

Therefore, for any given $\gamma > 0$, there exists $\delta > 0$ such that, for any $\epsilon_1, \epsilon_2 \in [0,1]$ satisfying $|\epsilon_1 - \epsilon_2| < \delta$,

$$|H(\epsilon_1) - H(\epsilon_2)| < \gamma.$$

This implies that $H(\epsilon) \in C([0,1],\mathbb{R})$.

Then we consider the continuity of $G(x,\epsilon)$. Note that $c(x)$ is continuous. Then, for the above δ, there exists $\delta_1 \in (0,\delta)$ such that, for any $x_1, x_2 \in \overline{\Omega}$ satisfying $|x_1 - x_2| < \delta_1$,

$$|c(x_1) - c(x_2)| < \delta/2.$$

Clearly, if $|\epsilon_1 - \epsilon_2| < \delta/2$ and $|x_1 - x_2| < \delta_1$, then

$$|c(x_2) + \epsilon_2 - c(x_1) - \epsilon_1| < \delta \ \text{ and } \ |c(x_2) - \epsilon_2 - c(x_1) + \epsilon_1| < \delta. \tag{5.2}$$

Choose $(x_1,\epsilon_1), (x_2,\epsilon_2) \in \overline{\Omega} \times [0,1]$ satisfying

$$|x_1 - x_2|,\ |\epsilon_1 - \epsilon_2| < \delta_2,$$

where $\delta_2 := \min\{\delta/2, \delta_1\}$. Clearly, there exists $u_1 \in [c(x_1) - \epsilon_1, c(x_1) + \epsilon_1]$ such that $G(x_1,\epsilon_1) = f(x_1,u_1)$. Then we claim that

$$G(x_1,\epsilon_1) < G(x_2,\epsilon_2) + \gamma,$$

and the proof is divided into two cases.

Case 1. $u_1 \in [c(x_2) - \epsilon_2, c(x_2) + \epsilon_2]$.

Since $|x_1 - x_2| < \delta_2 < \delta$, it follows from Eq. (5.1) that

$$G(x_1,\epsilon_1) = f(x_1,u_1) < f(x_2,u_1) + \gamma \le G(x_2,\epsilon_2) + \gamma.$$

Case 2. $u_1 \notin [c(x_2) - \epsilon_2, c(x_2) + \epsilon_2]$.

Then $u_1 > c(x_2) + \epsilon_2$ or $u_1 < c(x_2) - \epsilon_2$. We only consider the case of $u_1 > c(x_2) + \epsilon_2$, and the other case could be proved similarly. Then $c(x_2) + \epsilon_2 < u_1 \le c(x_1) + \epsilon_1$, This, combined with Eq. (5.2), implies that $|c(x_2) + \epsilon_2 - u_1| < \delta$. Then it follows from Eq. (5.1) that

$$G(x_1,\epsilon_1) = f(x_1,u_1) < f\left(x_2, c(x_2) + \epsilon_2\right) + \gamma \le G(x_2,\epsilon_2) + \gamma.$$

Then exchanging the positions of (x_1,ϵ_1) and (x_2,ϵ_2), we also have

$$G(x_2,\epsilon_2) < G(x_1,\epsilon_1) + \gamma.$$

This implies that for any given $\gamma > 0$, there exists $\delta_2 > 0$ such that, for any

$$(x_1, \epsilon_1), (x_2, \epsilon_2) \in \overline{\Omega} \times [0, 1]$$

satisfying $|x_1 - x_2| < \delta_2$ and $|\epsilon_1 - \epsilon_2| < \delta_2$,

$$|G(x_1, \epsilon_1) - G(x_2, \epsilon_2)| \leq \gamma. \tag{5.3}$$

This completes the proof. □

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