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Approximation Logic and Strong Bunge Algebra

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Abstract In this paper we give an axiom system of a logic which we call an approximation logic (AL), whose Lindenbaum-Tarski algebra is a strong Bunge algebra (or simply s-Bunge algebra), and show that

1. For every s-Bunge algebra B, a quotient algebra B^* by a maximal filter is isomorphic to the simplest nontrivial s-Bunge algebra $\Omega = \{0, a, 1\}$;
2. The Lindenbaum algebra of AL is an s-Bunge algebra;
3. AL is complete;
4. AL is decidable.

1 Introduction Marquis [2] investigated the relation between approximations and truth values. He considered Bunge algebras as representing approximations and tried to axiomatize the corresponding logic to Bunge algebras. Unfortunately he did not succeed. The following questions were left open:

- Q1. He introduced a concept of an implication $\rightarrow$ to Bunge algebras as $x \rightarrow y = \neg x \vee y$, but for this definition, he could not obtain the axiomatization of the corresponding logic;
- Q2. There are formulas A and B such that the truth values $v_f(A)$ and $v_f(B)$ are not equal even if they are equivalent in the logical system which he gives.

In this paper we give a different definition of an implication $x \rightarrow y$ axiomatically and solve these questions. First of all, we shall define a strong Bunge algebra (or simply s-Bunge algebra) which has a different implication operator "$\rightarrow$" from that of [2]. As special cases these algebras include Boolean and original Bunge algebras in [2] but not Heyting ones. For if we interpret Nx as a pseudo-complement element of x in the Heyting algebra $[0, 1]$, that is, $Nx = x \rightarrow 0$, then $1/2 \rightarrow 1/2 = 1$ but $N(1/2) \vee 1/2 = 1/2$. This means that in general the condition (I2) does not hold in Heyting algebras.

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Moreover we give an axiomatization of the logic which we call an approximation logic, or simply AL, whose Lindenbaum-Tarski algebras are s-Bunge algebras. We show the completeness theorem of the logic in terms of those algebras.

In the following we prove that

1. For every s-Bunge algebra B, a quotient algebra B^* by a maximal filter is isomorphic to the simplest nontrivial s-Bunge algebra $\Omega = \{0, a, 1\}$;
2. The Lindenbaum algebra of AL is an s-Bunge algebra;
3. For every formula A, A is provable in AL if and only if $\tau(A) = 1$ for any valuation function τ;
4. AL is decidable.

2 Strong Bunge algebras In this section we define a strong Bunge algebra (simply s-Bunge algebra) according to [2]. By an s-Bunge algebra, we mean the algebra $B = (B, \wedge, \vee, \to, N, 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$ such that

1. $(B, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
2. $N : B \to B$ is a map satisfying the following conditions:

   (N1) $x \vee Nx = 1$,

   (N2) $x \vee y = 1 \Rightarrow Nx \leq y$

   (N3) $N(x \wedge y) = Nx \vee Ny$

   (N4) $N(x \vee y) = Nx \wedge Ny$

3. the implication $\to$ satisfies

   (I1) $x \wedge y \leq z \Rightarrow y \leq x \to z$

   (I2) $x \to y \leq Nx \vee y$

   (I3) $Nx \wedge N(x \to y) \wedge y = 0$

   (I4) $x \wedge Nx \leq x \to y$

   (I5) $x \wedge Ny \leq N(x \to y) \vee y$

   (I6) $(x \to y) \wedge N(x \to y) \leq x$

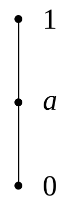
   (I7) $(x \to z) \wedge (y \to z) = x \vee y \to z$

   (I8) $(z \to x) \wedge (z \to y) = z \to x \wedge y$

The element Nx is regarded as a sup-complement of an element x in the Bunge algebra as in [2]. Bunge algebras without implication in [2] are considered as the same as our algebras satisfying only the conditions (1) and (2). For generality, we shall take N as a sup-complement operator, that is, N is a map from an s-Bunge algebra to itself satisfying the conditions (N1) – (N4). Moreover we propose an implication operator '$\to$' which is different from that of [2]. It is easy to show that, in the case of Boolean algebras, Nx is the complement of x, $Nx = x \to 0$, and $x \to y = Nx \vee y$.

Example 2.1 As models of Bunge algebras we list $\Omega = \{0, a, 1\}$, $\Omega(4)$, and $\Omega(6)$. Ω is the simplest nontrivial s-Bunge algebra.

Ω



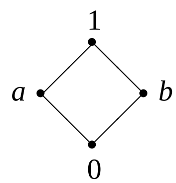
$Na = 1$

$x \to y$

$x \backslash y$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

Figure 1

$\Omega(4)$

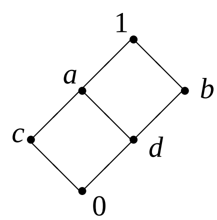


$Na = a$
$Nb = b$

$x \backslash y$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Figure 2

$\Omega(6)$



$Na = b$
$Nb = c$
$Nc = b$
$Nd = 1$

$x \backslash y$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	b	a	b	1
b	c	a	1	c	a	1
c	b	1	b	1	b	1
d	a	1	1	a	1	1
1	0	a	b	c	d	1

Figure 3

Remark 2.2

1. By (I1), we have that $x \to x = 1$, $y \le x \to y$, and $x \le y$ implies $x \to y = 1$ for every x and y.
2. The truth table of $x \to y$ are uniquely determined as above by our definitions of implication.
3. In the s-Bunge algebra $\Omega = (\{0, a, 1\}, \wedge, \vee, N, \to, 0, 1)$, if we define $x^* = x \to 0$ then the structure $(\{0, a, 1\}, \wedge, \vee, {}^*, \to, 0, 1)$ becomes the Kleene algebra 3 in Kondo [1]. Conversely in the Kleene algebra $3 = (\{0, 1/2, 1\}, \wedge, \vee, {}^*, \to, 0, 1)$ the definition $Nx = x^* \to x$ yields that $(\{0, 1/2, 1\}, \wedge, \vee, N, \to, 0, 1)$ is the s-Bunge algebra.

In the following we shall prove the representation theorem of s-Bunge algebras which is one of the main theorems of this paper: for every s-Bunge algebra B, the quotient algebra B^* by a maximal filter is isomorphic to the simplest s-Bunge algebra Ω.

Let B be any s-Bunge algebra. A nonempty subset F of B is called a filter when it satisfies the conditions:

(f1) $x, y \in F$ imply $x \wedge y \in F$;

(f2) $x \in F$ and $x \le y$ imply $y \in F$.

A filter F is called proper when it is a proper subset of B. We define two kinds of filters of B. By a maximal filter F, we mean the proper filter F such that $F \subseteq G$ implies $F = G$ for any proper filter G. A proper filter F is called prime if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for every $x, y \in B$.

It is easy to show that the next lemmas hold for any distributive lattice, so we omit their proofs.

Lemma 2.3 *If $x \in B$ and $x \neq 0$, then there is a maximal filter M of B such that $x \in M$.*

Lemma 2.4 *If M is a maximal filter of B, then it is a prime filter.*

Lemma 2.5 *If M is a maximal filter and $x \notin M$, then there is an element $u \in M$ such that $x \wedge u = 0$.*

We note that $x \notin M$ implies $Nx \in M$, provided that M is prime. In the following, let M be a maximal filter of B.

Now we define a relation $\sim$ as follows: for $x, y \in B$,

$$x \sim y \iff \text{there exists } m \in M \text{ such that: } x \wedge m = y \wedge m,\ Nx \wedge m = Ny \wedge m, \text{ and } N^2x \wedge m = N^2y \wedge m.$$

The relation $\sim$ is obtained alternatively as follows.

Lemma 2.6 *$x \sim y$ iff*

$$\begin{aligned} x \in M &\Leftrightarrow y \in M, \\ Nx \in M &\Leftrightarrow Ny \in M, \text{ and} \\ N^2x \in M &\Leftrightarrow N^2y \in M. \end{aligned}$$

Lemma 2.7 *The relation $\sim$ is a congruence relation on B.*

Proof: It suffices to show that

1. $x \sim p$ and $y \sim q$ implies $x \wedge y \sim p \wedge q$;
2. $x \sim p$ and $y \sim q$ implies $x \vee y \sim p \vee q$;
3. $x \sim p$ implies $Nx \sim Np$;
4. $x \sim p$ and $y \sim q$ implies $x \to y \sim p \to q$;

It is clear by Lemma 2.6 that the conditions (1) – (3) hold, so we consider the case of (4).

Suppose that $x \sim p$ and $y \sim q$. It is sufficient to show that

$$\begin{aligned} x \to y \in M &\quad \text{iff} \quad p \to q \in M, \\ N(x \to y) \in M &\quad \text{iff} \quad N(p \to q) \in M, \text{ and} \\ N^2(x \to y) \in M &\quad \text{iff} \quad N^2(p \to q) \in M. \end{aligned}$$

We show only that the left-hand statement implies the right-hand one in each case.

Case 1: $x \to y \in M$: We need to show that $p \to q \in M$. Since $x \to y \leq Nx \vee y$ and M is maximal, we have two cases, $Nx \in M$ or $y \in M$.

Subcase 1: $Nx \in M$ (i.e., $Np \in M$): There are two subcases, $x \in M$ or $x \notin M$. If $x \in M$ (i.e., $p \in M$), then we have $p \to q \in M$ by $p \wedge Np \leq p \to q$. The other case, $x \notin M$, yields $p \notin M$. There is an element $u \in M$ such that $p \wedge u = 0 \leq q$. It follows that $u \leq p \to q \in M$.

Subcase 2: $y \in M$: Since $q \in M$ and $q \leq p \to q$, we have $p \to q \in M$.

Thus $x \to y \in M$ implies $p \to q \in M$.

Case 2: $N(x \to y) \in M$: We have to prove that $N(p \to q) \in M$. Since $N(x \to y) \leq Ny$, we obtain that $Ny \in M$ and $Nq \in M$. Condition (I2) yields $N^2p \wedge Nq \leq N(p \to q)$, so there are two subcases, $N^2p \in M$ or $N^2p \notin M$.

Subcase 1: From $N^2p \in M$ it follows $N(p \to q) \in M$ by $Nq \in M$.

Subcase 2: $N^2p \notin M$: Since $Np \vee N^2p = 1 \in M$, we have $Np \in M$ and hence $Nx \in M$. By (I3) we have $y \notin M$ (i.e., $q \notin M$). If $p \in M$ then we have $N(p \to q) \in M$. In case of $p \notin M$ (hence $x \notin M$), there exists u in M such that $x \wedge u = 0 \leq y$. This means that $x \to y \in M$. It follows that $(x \to y) \wedge N(x \to y) \leq x \in M$. But this is a contradiction.

Thus $N(x \to y) \in M$ implies $N(p \to q) \in M$.

Case 3: $N^2(x \to y) \in M$: By (I2) we get that $N^2(x \to y) \leq Nx \vee N^2y$ and that $Nx \in M$ or $N^2y \in M$. The case of $N^2y \in M$ yields $N^2q \in M$ and $N^2(p \to q) \in M$ by $N^2q \leq N^2(p \to q)$. In the case of $Nx \in M$, we have $Np \in M$ and $N^2p \notin M$ by $Np \wedge N^2p = 0 \notin M$. Since $N^2p \vee N^2(p \to q) \vee Nq = 1 \in M$, this means that $N^2(p \to q) \in M$ or $Nq \in M$. It is sufficient to consider the case of $Nq \in M$ (hence $Ny \in M$). There are two subcases, $p \in M$ or $p \notin M$.

Subcase 1: Since $x \in M$ and $x \wedge Ny \leq N(x \to y) \vee y \in M$, it follows that $N(x \to y) \in M$ or $y \in M$. But $N^2(x \to y) \in M$ yields that $y \in M$ and hence $q \in M$. Now, the assumption $Np, q \in M$ and (I3) imply that $N(p \to q) \notin M$. Therefore $N^2(p \to q) \in M$.

Subcase 2: Since $p \notin M$, we have $p \to q \in M$. It follows from $(p \to q) \wedge N(p \to q) \leq p \notin M$ that $N(p \to q) \notin M$ and $N^2(p \to q) \in M$.

Thus we can show that $N^2(x \to y) \in M$ implies $N^2(p \to q) \in M$. Therefore the relation $\sim$ is a congruence relation on B, provided that M is a maximal filter. $\square$

Let M be a maximal filter of B and $[x] = \{y \in B \mid x \sim y\}$ be the equivalence class of x and $B/\!\sim\, = \{[x] \mid x \in B\}$ be the set of all equivalence classes. The congruence of the relation $\sim$ entails that we can consistently define operations $\wedge$, $\vee$, N, and $\to$ on $B/\!\sim$ as follows:

$$\begin{aligned} [x] \wedge [y] &= [x \wedge y] \\ [x] \vee [y] &= [x \vee y] \\ N[x] &= [Nx] \\ [x] \to [y] &= [x \to y]. \end{aligned}$$

By simple calculation (cf. [1]) an s-Bunge algebra B is the direct union of the following three subsets B_j ($j \in \Omega$):

$$\begin{aligned} B_0 &= \{x \in B \mid x \notin M,\ Nx \in M,\ N^2x \notin M\} \\ B_a &= \{x \in B \mid x \in M,\ Nx \in M,\ N^2x \notin M\} \\ B_1 &= \{x \in B \mid x \in M,\ Nx \notin M,\ N^2x \in M\}. \end{aligned}$$

Since M is a maximal filter, we can prove that the subset B_j equals some equivalence class.

Theorem 2.8

$$\begin{aligned} x \in B_0 &\iff x \sim 0 \\ x \in B_a &\iff x \not\sim 0 \textit{ and } Nx \sim 1 \\ x \in B_1 &\iff x \sim 1 \end{aligned}$$

Proof: We show only the case of B_a. If $x \in B_a$, then we have $x \in M$, $Nx \in M$, and $N^2x \notin M$. Since M is maximal, there is u in M such that $N^2x \wedge u = 0$. We put $\alpha = x \wedge Nx \wedge u \in M$. For that element α we obtain that

1. $Nx \wedge \alpha = \alpha = 1 \wedge \alpha$;
2. $N^2x \wedge \alpha = 0 = N1 \wedge \alpha$;
3. $N^3x \wedge \alpha = Nx \wedge \alpha = \alpha = N^21 \wedge \alpha$.

This means that $Nx \sim 1$.

If $x \sim 0$, then there is v in M such that $x \wedge v = 0$, $Nx \wedge v = N0 \wedge v = v$, and $N^2x \wedge v = 0$. Since $x \in M$, we get that $x \wedge v = 0 \in M$. But this is a contradiction. Thus $x \not\sim 0$.

Conversely, we suppose that $x \not\sim 0$ and $Nx \sim 1$. By definition of $\sim$, there exists an element $f \in M$ such that

$$\begin{aligned} Nx \wedge f &= 1 \wedge f = f, \\ N^2x \wedge f &= N1 \wedge f = 0, \text{ and} \\ N^3x \wedge f &= N^21 \wedge f = f. \end{aligned}$$

It follows that $Nx \in M$ and hence that $N^2x \notin M$ by $f \leq Nx$. If $x \notin M$, then $x \wedge g = 0$ for some $g \in M$. Put $\beta = f \wedge g (\in M)$. For that element β we get that $x \sim 0$ by simple calculation. This contradicts the assumption. Hence $x \in M$. And so $x \in B_a$. □

The next theorem is a fact of universal algebras.

Theorem 2.9 *Let B be an s-Bunge algebra and M be a maximal filter of B. A quotient algebra $B/{\sim} = (B/{\sim}, \wedge, \vee, N, \rightarrow, [0], [1])$ by M is an s-Bunge algebra, and it is isomorphic to the simplest nontrivial s-Bunge algebra Ω.*

Proof: The map $\xi : B/{\sim} \longrightarrow \Omega$ defined by $\xi([x]) = j$ when $[x] \in B_j$ gives the desired result. □

***3 Approximation logic* AL** In this section we shall define an approximation logic AL. The language of AL is a countable set of propositional variables $p_1, p_2, \ldots, p_n, \ldots$, propositional constant f, and logical symbols $\wedge, \vee, \neg$, and $\rightarrow$. We denote the set of propositional variables and f by Π, that is, $\Pi = \{f, p_1, p_2, \ldots, p_n, \ldots\}$. The formulas of AL are defined as usual. Let $A, B, C, \ldots$ be arbitrary formulas of AL. In the following we list an axiom system of AL.

Axioms

A1 $A \wedge B \to A$
A2 $A \to A \vee B$
A3 $A \wedge B \to B \wedge A$
A4 $A \vee B \to B \vee A$
A5 $A \vee \neg A$
A6 $A \to A \wedge A$
A7 $\neg A \wedge \neg B \to \neg(A \vee B)$
A8 $\neg(A \wedge B) \to \neg A \vee \neg B$
A9 $(A \to B) \to (A \wedge C \to B \wedge C)$
A10 $(A \to B) \to (A \vee C \to B \vee C)$
A11 $f \to A$
A12 $(A \to B) \to ((B \to C) \to (A \to C))$
A13 $A \to (B \to A)$
A14 $(A \to (B \to C)) \to (B \to (A \to C))$
A15 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
A16 $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$
A17 $(A \wedge B \to f) \to (A \to (B \to f))$
A18 $(A \to B) \to (\neg A \vee B)$
A19 $\neg A \wedge \neg(A \to B) \wedge B \to f$
A20 $A \wedge \neg A \to (A \to B)$
A21 $A \wedge \neg B \to \neg(A \to B) \vee B$
A22 $(A \to f) \vee B \to (A \to B)$
A23 $(A \to B) \wedge \neg(A \to B) \to A$

Rules of inference

R1 B is deduced from A and $A \to B$ (modus ponens, MP);
R2 $\neg B \to \neg A$ is deduced from $A \to B$;
R3 $\neg A \to B$ is deduced from $A \vee B$.

Let A be a formula of AL. By $\vdash_{\mathrm{AL}} A$ we mean that there is a sequence of formulas $A_1, A_2, \ldots, A_n$ of AL such that:

1. $A = A_n$
2. For every A_i, it is an axiom or it is deduced from A_j and A_k $(j, k < i)$ by rules of inference.

We say that A is provable in AL when $\vdash_{\mathrm{AL}} A$. If no confusion arises, we denote it simply by $\vdash A$.

It is easy to show the next lemmas, so we omit their proofs.

Lemma 3.1 *(1) If* $\vdash A \to B$ *and* $\vdash B \to C$, *then we have* $\vdash A \to C$.
(2) For every formula A *of* AL, *we have that* $\vdash A \to (f \to f)$ *and* $\vdash f \to A$.

We denote $f \to f$ by t. A function $\tau : \Pi \longrightarrow \Omega$ such that $\tau(f) = 0$ is called a valuation function. The domain of the valuation function can be extended uniquely to the set Φ of all formulas of AL as follows:

$$\begin{aligned} \tau(A \wedge B) &= \tau(A) \wedge \tau(B) \\ \tau(A \vee B) &= \tau(A) \vee \tau(B) \\ \tau(\neg A) &= N\tau(A) \\ \tau(A \to B) &= \tau(A) \to \tau(B). \end{aligned}$$

Henceforth we use the same symbol τ for the extended valuation function.

We can show that the approximation logic AL is sound for the Bunge algebra Ω, that is, if $\vdash_{\mathrm{AL}} A$ then $\tau(A) = 1$ for any valuation function τ.

Theorem 3.2 *Let A be an arbitrary formula of* AL. *If $\vdash_{\mathrm{AL}} A$ then $\tau(A) = 1$ for every valuation function τ.*

Proof: By induction on the construction of a proof. It suffices to show that $\tau(X) = 1$ for every axiom X and that the rules of inference preserve validity. We show only that the axiom (A18) $(A \to B) \to \neg A \vee B$ is valid. If it is not valid, then there is a valuation function τ such that $\tau(A \to B) = 1$ and $\tau(\neg A \vee B) = 0$, $\tau(A \to B) = 1$ and $\tau(\neg A \vee B) = a$, or $\tau(A \to B) = a$ and $\tau(\neg A \vee B) = 0$ by Figure 1 in Section 2. We shall show that in each case there is a contradiction. For the sake of simplicity we consider only the case of $\tau(A \to B) = a$ and $\tau(\neg A \vee B) = 0$. Since $\tau(\neg A \vee B) = 0$, we have $\tau(\neg A) = \tau(B) = 0$ and so $\tau(A) = 1$. This means that $\tau(A \to B) = 1 \to 0 = 0 \neq a$. This is a contradiction. The other cases are proved similarly. Thus the axiom (A18) is valid. □

As corollaries to the theorem we have following.

Corollary 3.3 AL *is consistent.*

Proof: Since $\tau(f) = 0$, the formula f is not provable in AL. Thus the approximation logic is consistent. □

Corollary 3.4 *The approximation logic is different from the classical propositional logic (CPL) and the intuitionistic propositional logic (IPL).*

Proof: If we think of a valuation function τ such that $\tau(p) = a$ for any propositional variable p, then we have that $\tau(p \to \neg\neg p) = a \to N(Na) = a \to 0 = a \neq 1$, and hence that the formula $p \to \neg\neg p$ is not provable in AL. Thus the approximation logic is different from CPL. Next, the formula $\neg\neg A \to A$ is not provable in IPL in general, but it is provable in AL. Thus IPL is not equal to AL. □

4 Completeness theorem In this section we shall establish the completeness theorem of the approximation logic AL, and it is the main theorem of this paper. The completeness theorem of AL means that a formula A is provable in AL if $\tau(A) = 1$ for any valuation function τ. In order to show the theorem, we consider the Lindenbaum-Tarski algebra of AL and investigate the property of that algebra.

We introduce the relation $\equiv$ on Φ as follows. For $A, B \in \Phi$,

$$A \equiv B \text{ iff } \vdash_{\mathrm{AL}} A \to B \text{ and } \vdash_{\mathrm{AL}} B \to A.$$

Lemma 4.1 *The relation* $\equiv$ *is a congruence relation on* Φ.

Proof: We show only that the relation $\equiv$ satisfies the following conditions: if $A \equiv X$ and $B \equiv Y$, then

(a) $A \wedge B \equiv X \wedge Y$
(b) $A \vee B \equiv X \vee Y$
(c) $\neg A \equiv \neg B$
(d) $A \rightarrow B \equiv X \rightarrow Y$

It is evident that conditions (a), (b), and (c) hold from axioms (A9), (A10), and the inference rule (R2). We prove that the condition (d) holds. Suppose that $A \equiv X$ and $B \equiv Y$. For condition (d), since $\vdash (B \rightarrow Y) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow Y))$ by (A12), (A14), and (MP), we have $\vdash (A \rightarrow B) \rightarrow (A \rightarrow Y)$ by assumption $\vdash B \rightarrow Y$. Similarly it follows that $\vdash (A \rightarrow Y) \rightarrow ((X \rightarrow A) \rightarrow (X \rightarrow Y))$. Thus, we obtain that $\vdash (A \rightarrow B) \rightarrow (X \rightarrow Y)$. A similar argument yields the converse $\vdash (X \rightarrow Y) \rightarrow (A \rightarrow B)$.

Hence the relation $\equiv$ is the congruence relation. □

We define the quotient set L^* of Φ by the congruence relation $\equiv$. That is, we set $L^* = \{[A] \mid A \in \Phi\}$, where $[A] = \{X \in \Phi \mid A \equiv X\}$. We introduce an order relation $\sqsubseteq$ on L^* as follows: for any $[A], [B] \in L^*$,

$$[A] \sqsubseteq [B] \text{ iff } \vdash_{\mathrm{AL}} A \rightarrow B.$$

Since the relation $\equiv$ is congruent, it is clear that the definition of $\sqsubseteq$ is well-defined and that the relation $\sqsubseteq$ is a partial order. Concerning to this order we have the following.

Lemma 4.2 $inf\{[A], [B]\} = [A \wedge B],\ sup\{[A], [B]\} = [A \vee B]$

Proof: We shall show the first case for the sake of simplicity. The second case can be proved analogously.

Since $\vdash A \wedge B \rightarrow A$ and $\vdash A \wedge B \rightarrow B$, we obtain $[A \wedge B] \sqsubseteq [A], [B]$. For any $[C]$ such that $[C] \sqsubseteq [A], [B]$, since $\vdash C \rightarrow A$ and $\vdash C \rightarrow B$, it follows that $\vdash C \wedge B \rightarrow A \wedge B$ by $\vdash (C \rightarrow A) \rightarrow ((C \wedge B) \rightarrow (A \wedge B))$. Thus it means $\vdash B \wedge C \rightarrow A \wedge B$. On the other hand, $\vdash (C \rightarrow B) \rightarrow ((C \wedge C) \rightarrow (B \wedge C))$ and $\vdash C \rightarrow B$ yield $\vdash C \wedge C \rightarrow B \wedge C$. So we have $\vdash C \rightarrow B \wedge C$. These mean that $\vdash C \rightarrow A \wedge B$ and hence that $[C] \sqsubseteq [A \wedge B]$. Thus we have $inf\{[A], [B]\} = [A \wedge B]$. □

By the lemma we can define the operations $\sqcap$ and $\sqcup$ respectively by

$$[A] \sqcap [B] = inf\{[A], [B]\} = [A \wedge B]$$
$$[A] \sqcup [B] = sup\{[A], [B]\} = [A \vee B].$$

It is easy to show that the structure $(L^*, \sqcap, \sqcup)$ is a lattice. Moreover, if we put $[t] = 1, [f] = 0, N[A] = [\neg A]$, and $[A] \Rightarrow [B] = [A \rightarrow B]$, then the axioms of AL assures that the structure $(L^*, \sqcap, \sqcup, N, \Rightarrow, 0, 1)$ is an s-Bunge algebra. The structure is called a Lindenbaum-Tarski algebra of AL. Hence we have the following theorem.

Theorem 4.3 *The Lindenbaum-Tarski algebra* L^* *of the approximation logic* AL *is an s-Bunge algebra.*

We have an important lemma concerning the algebra L^*.

Lemma 4.4 *For every formula A, $\vdash_{\mathrm{AL}} A$ iff $[A] = 1$ in L^*.*

Proof: Suppose that $\vdash A$. Since $A \to (t \to A)$ is provable in AL, we get that $\vdash t \to A$, that is, $[A] = 1$. Conversely if we assume that $[A] = 1$ then it follows $\vdash t \to A$ by definition. Thus we have $\vdash A$ by $\vdash t$. □

Now we shall prove the completeness theorem of AL. In order to show that, it suffices to indicate the existence of a valuation function τ such that $\tau(A) \neq 1$ if A is not provable in AL. Suppose that a formula A is not provable in AL. In the Lindenbaum-Tarski algebra L^* of AL, we have $[A] \neq 1$ by the lemma above. It means that $N[A] \neq 0$. By Lemma 2.3, there is a maximal filter M^* in L^* such that $N[A] \in M^*$. Using the filter M^* we define a valuation function τ. For any propositional variable p, we put

$$\tau(p) = \begin{cases} 1 & \text{if } [p] \in M^*, N[p] \notin M^*, \text{ and } N^2[p] \in M^* \\ a & \text{if } [p] \in M^*, N[p] \in M^*, \text{ and } N^2[p] \notin M^* \\ 0 & \text{if } [p] \notin M^*, N[p] \in M^*, \text{ and } N^2[p] \notin M^*. \end{cases}$$

As to that function τ, we can show the following lemma.

Lemma 4.5 *For any formula $X \in \Phi$,*

$$\tau(X) = \begin{cases} 1 & \textit{if } [X] \in M^*, N[X] \notin M^*, \textit{ and } N^2[X] \in M^* \\ a & \textit{if } [X] \in M^*, N[X] \in M^*, \textit{ and } N^2[X] \notin M^* \\ 0 & \textit{if } [X] \notin M^*, N[X] \in M^*, \textit{ and } N^2[X] \notin M^*. \end{cases}$$

Proof: It suffices to show that τ satisfies the following: for arbitrary formulas X and Y,

1. $\tau([X] \wedge [Y]) = \tau([X]) \wedge \tau([Y])$
2. $\tau([X] \vee [Y]) = \tau([X]) \vee \tau([Y])$
3. $\tau(N[X]) = N\tau([X])$
4. $\tau([X] \to [Y]) = \tau([X]) \to \tau([Y])$

For the sake of simplicity, we show only cases (3) and (4). Let $x = [X]$ and $y = [Y]$.

Case 3: $\tau([X]) = a$: It suffices to show that $\tau(N[X]) = 1$, that is, $Nx \in B_1$. By assumption we have $x \notin M^*$, $Nx \in M^*$, and $N^2x \notin M^*$. Clearly we get that $Nx \in M^*$, $N^2x \notin M^*$, and $N^3x = Nx \in M^*$. This means that $Nx \in B_1$. Th other cases are proved similarly.

Case 4:

Subcase 1: $\tau([X]) = a, \tau([Y]) = 0$: It is sufficient to prove that $x \to y \in B_a$, that is $x \to y$, $N(x \to y) \in M^*$, and $N^2(x \to y) \notin M^*$. By assumption we have $x \in M^*$, $Nx \in M^*$, $N^2x \notin M^*$, $y \notin M^*$, $Ny \in M^*$, and $N^2y \notin M^*$. We then have $x \to y \in M^*$, because x and Nx are in M^* and $x \wedge Nx \leq x \to y$. From $x \wedge Ny \leq N(x \to y) \vee y$, we get that $N(x \to y) \in M^*$ or $y \in M^*$. Since $y \notin M^*$, it follows that $N(x \to y) \in M^*$. Hence $N^2(x \to y) \notin M^*$. These entail that $x \to y \in B_a$.

Subcase 2: $\tau([X]) = a, \tau([Y]) = a$: In this case we have $x, Nx, y, Ny \in M^*$ and $N^2x, N^2y \notin M^*$ by definition. We can conclude that $x \to y$, $N^2(x \to y) \in M^*$ and $N(x \to y) \notin M^*$ because it is evident that $x \to y \in M^*$ by $y \leq x \to y$. And because $Nx \wedge N(x \to y) \wedge y = 0 \notin M^*$, the fact that Nx and y are in M^* implies $N(x \to$

$y) \notin M^*$ and hence $N^2(x \to y) \in M^*$. This yields that $x \to y \in B_1$. Thus we have $\tau([X] \to [Y]) = 1$.

Subcase 3: $\tau([X]) = 1, \tau([Y]) = a$: It suffices to indicate that $x \to y \in B_a$, that is, $x \to y \in M^*$, $N(x \to y) \in M^*$, and $N^2(x \to y) \notin M^*$. By definition, we have $x, N^2x, y, Ny \in M^*$ and $Nx, N^2y \notin M^*$. Since $y \in M^*$ and $y \leq x \to y$, we get that $x \to y \in M^*$. The condition $x \to y \leq Nx \vee y$ implies $N^2x \wedge Ny \leq N(x \to y)$. Therefore we also have $N(x \to y) \in M^*$ by assumption. It follows that $N^2(x \to y) \notin M^*$. Thus we obtain $x \to y \in B_a$.

The other cases are proved similarly.

This completes the proof of the lemma. □

Since $N[A] \in M^*$, it follows that $\tau(A) \neq 1$ by the lemma. Hence we have the completeness theorem of AL.

Theorem 4.6 *For any formula A,* $\vdash A$ *iff* $\tau(A) = 1$ *for every valuation function* τ.

It turns out from the theorem that it is sufficient to calculate the value of $\tau(A)$ whether the formula A is provable or not in AL. Since any formula has at most finite numbers of propositional variables, say n, the possible values of the n-tuple of the propositional variables in that formula are finite (at most 3^n). Thus we can establish the following.

Theorem 4.7 *The approximation logic* AL *is decidable.*

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