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EDGE-CONNECTIVITY OF STRONG PRODUCTS OF GRAPHS

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Abstract

The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 is the graph with $V(G_1) \times V(G_2)$ as the vertex set, and two distinct vertices (x_1, x_2) and (y_1, y_2) are adjacent whenever for each $i \in \{1, 2\}$ either $x_i = y_i$ or $x_i y_i \in E(G_i)$. In this note we show that for two connected graphs G_1 and G_2 the edge-connectivity $\lambda(G_1 \boxtimes G_2)$ equals $\min\{\delta(G_1 \boxtimes G_2), \lambda(G_1)(|V(G_2)| + 2|E(G_2)|), \lambda(G_2)(|V(G_1)| + 2|E(G_1)|)\}$. In addition, we fully describe the structure of possible minimum edge cut sets in strong products of graphs.

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1. Introduction

The vertex and the edge connectivity of a graph are among the most natural and well-studied graph invariants that found applications in communication networks and elsewhere. For a connected graph G, a set S of edges of G is called *separating* if $G - S$ (the graph obtained from G by deletion of edges from S) is not connected. The *edge-connectivity* $\lambda(G)$ of a graph G is the cardinality of a minimum separating set in G. Clearly $\lambda(G) \leq \delta(G)$ where $\delta(G)$ denotes the minimum degree of vertices in G. We will say that a separating set S in a graph G is a λ-*set* in G if $|S| = \lambda(G)$.

The strong product is one of the four standard graph products [1]. It is often the case that graph invariants of a product of graphs can be expressed in terms of some invariants of their factors which are smaller and thus easier to consider. The vertex and the edge connectivity of the Cartesian product of graphs have already been determined [2, 4]. Concerning the strong product of graphs, so far only the vertex connectivity was established [3]. Our aim in this note is to exactly determine the edge-connectivity of the strong product of graphs.

It may not be surprising that the edge-connectivity of $G \boxtimes H$ is often equal to the minimum degree. However, this is not the only case which makes the formula more complex, and the proof more interesting. In addition, we also describe the structure of λ-sets in each of the cases that can appear.

Theorem 1. *Let G and H be connected graphs. Then*

$$\lambda(G\boxtimes H) = \min\{\delta(G\boxtimes H), \lambda(G)(|V(H)|+2|E(H)|), \lambda(H)(|V(G)|+2|E(G)|)\}.$$

Moreover, if S is a λ-set in $G \boxtimes H$ and C_1, C_2 are connected components of $(G \boxtimes H) - S$ then one of the following occurs:

(i) $C_1 = \{x\}$ *and* $C_2 = V(G) \times V(H) - \{x\}$ *for some* $x \in V(G \boxtimes H)$ *or*

(ii) $C_1 = C_1' \times V(H)$ *and* $C_2 = C_2' \times V(H)$ *or*

(iii) $C_1 = V(G) \times C_1''$ *and* $C_2 = V(G) \times C_2''$

where C_i' and C_i'' (for $i = 1, 2$) are connected components of $G - S'$ and $H - S''$ for some λ-sets S' and S'' in G and H, respectively.

The note is organized as follows. In the next section we prove some preliminary observations that are needed in the proof of the theorem. In the third section Theorem 1 is proved, and in the last section some examples are presented that show, each of the expressions in the formula can be achieved.

2. Key Observations

The following observation is straightforward.

Lemma 2. *Let S be a λ-set in a graph G. Then $G - S$ consists of two connected components.*

Let S be a λ-set of $G \boxtimes H$. By the lemma there are two connected components C_1 and C_2 of $G \boxtimes H - S$, and we will say that the vertices of C_1 are *black* and the vertices of C_2 *white*. Hence an edge is from S if and only if its endvertices are of different colors. The edges from S will be called *grey* edges.

Let $G \boxtimes H$ be the strong product of graphs G and H. For $v \in V(H)$ the set $G_v = \{(x, v) \mid x \in V(G)\}$ is called a *G-fiber*. It is clear that the subgraph of $G \boxtimes H$ induced by G_v is isomorphic to G. Analogously we define *H-fibers* that clearly induce subgraphs of $G \boxtimes H$ that are isomorphic to H. As usual, the projections p_G and p_H map vertices from $V(G \boxtimes H)$ to their first and second coordinates, respectively. The following lemma is one of the main keys in the proof.

Lemma 3. *Let G and H be nontrivial connected graphs and $S \subseteq E(G \boxtimes H)$ a λ-set in $G \boxtimes H$. Let C_1 and C_2 be connected components of $(G \boxtimes H) - S$ and suppose that $uv \in E(H)$ is an edge such that $G_u \cap C_i$ and $G_v \cap C_i$ are nonempty for $i = 1, 2$. Then there are at least $2\,\lambda(G)$ edges $e \in S$, such that one endvertex of e is in G_u and the other in G_v. Moreover, if there are exactly $2\,\lambda(G)$ such edges, then $p_G(G_u \cap C_i) = p_G(G_v \cap C_i)$ for $i = 1, 2$.*

Proof. We refer to edges with one endvertex in G_u and the other in G_v as the *cross edges*. We define the following sets

$$A = \{x \in V(G) \mid (x, u) \in C_1, (x, v) \in C_2\},$$
$$B = \{x \in V(G) \mid (x, u) \in C_2, (x, v) \in C_1\},$$
$$C = \{x \in V(G) \mid (x, u) \in C_1, (x, v) \in C_1\},$$
$$D = \{x \in V(G) \mid (x, u) \in C_2, (x, v) \in C_2\}.$$

Clearly the sets A, B, C and D are pairwise disjoint (and not necessarily nonempty) and their union is $V(G)$.

First suppose that there is no edge $ab \in E(G)$ such that $a \in A$ and $b \in B$. Let $S_v \subseteq S$ be the set of grey edges with endvertices in G_v (that are the edges with one endvertex in $C_1 \cap G_v$ and the other in $C_2 \cap G_v$). Similarly, let $S_u \subseteq S$ be the set of grey edges with endvertices in G_u. Since S is a separating set of $G \boxtimes H$, S_v (resp. S_u) must be a separating set of the subgraph induced by G_v (resp. G_u), and so $|S_v|, |S_u| \geq \lambda(G)$. Now we have the following observation (see Figure 1):

$$(x,v)(y,v) \in S_v \ \text{ and } \ (x,u)(y,u) \in S_u \iff xy \in E(G), x \in C, y \in D\,.$$

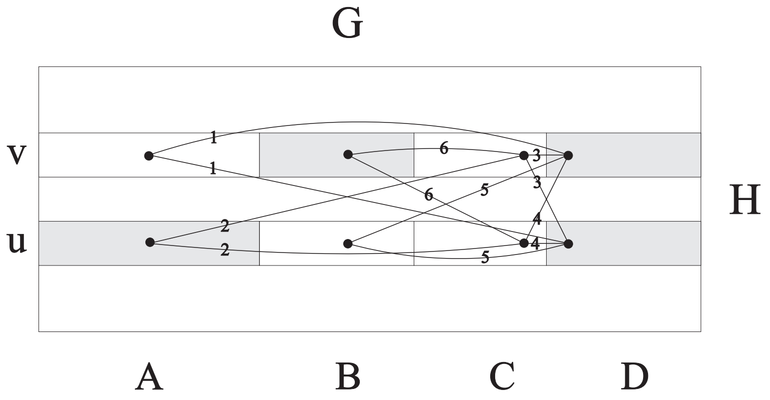

Figure 1. Case when there are no edges between A and B. The edges with both endvertices in a G-fiber and corresponding cross edges are marked with the same number.

Observe also that $(x,v)(y,v) \in S_v$ and $(x,u)(y,u) \in S_u$ implies $(x,v)(y,u) \in S$ and $(x,u)(y,v) \in S$. If $(x,v)(y,v) \in S_v$ and $(x,u)(y,u) \notin S_u$ then observe that either $(x,v)(y,u) \in S$ or $(x,u)(y,v) \in S$. Similarly if $(x,v)(y,v) \notin S_v$ and $(x,u)(y,u) \in S_u$ then either $(x,v)(y,u) \in S$ or $(x,u)(y,v) \in S$. We infer there are at least $|S_v| + |S_u|$ grey cross edges between fibers G_u and G_v. Since $|S_v|, |S_u| \geq \lambda(G)$, we find that there are at least $2\lambda(G)$ cross edges between G_u and G_v that are from S.

Also note that if A or B is nonempty, then there are additional cross edges in S, namely $(x,v)(x,u)$, where $x \in A \cup B$. Hence if the number of cross edges is exactly $2\lambda(G)$ then $A = B = \emptyset$. In other words, $p_G(G_u \cap C_i) = p_G(G_v \cap C_i)$ for $i = 1, 2$ which proves the claim in the last assertion of the lemma for this case.

Now suppose that there is at least one edge $ab \in E(G)$ such that $a \in A$ and $b \in B$. If there is exactly one such edge, then for the edges $(a, v)(b, v) \in S_v$ and $(a, u)(b, u) \in S_u$ we get two grey cross edges $(a, v)(a, u)$ and $(b, v)(b, u)$. For the other edges of S_v and S_u we argue similarly as above, and again derive there are at least $|S_v| + |S_u| \geq 2\lambda(G)$ grey cross edges between G_u and G_v.

Finally assume there is more than one edge $ab \in E(G)$ such that $a \in A$ and $b \in B$. In this case $|A| \geq 2$ or $|B| \geq 2$. Without loss of generality assume that $|A| \geq |B|$ (and thus $|A| \geq 2$). Let $a \in A$ be a vertex with minimum number of neighbors in $C \cup D$ among vertices from A (see Figure 2).

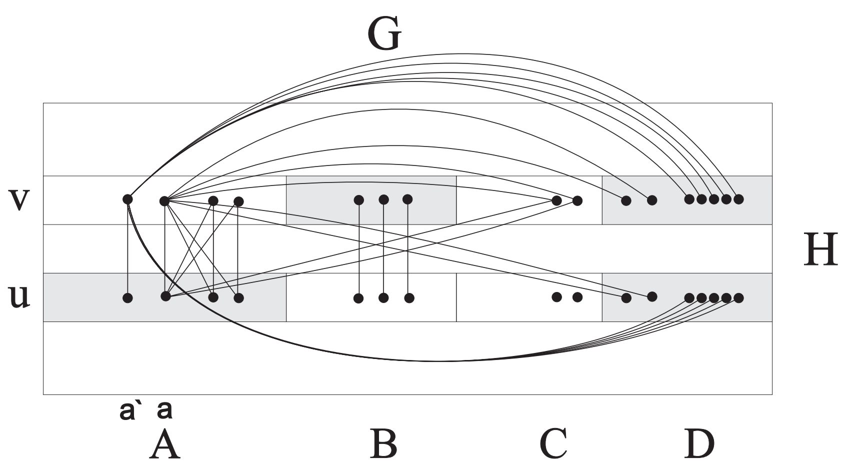

Figure 2. Case when there are edges between A and B.

We claim there are at least $2\deg_G(a)$ cross edges in S. For all neighbors $x \in A$ of a we get two grey cross edges, namely $(a, v)(x, u)$ and $(x, v)(a, u)$. The number of neighbors of a in B is at most $|B|$. Since $2|B| \leq |A| + |B|$ we find that there are at least $2|B|$ edges $(x, u)(x, v)$ in S, where $x \in A \cup B$ (note that these are also cross edges). For each neighbor $x \in C \cup D$ either $(a, v)(x, u)$ or $(x, v)(a, u)$ is in S. By the initial assumption about a, any other vertex $a' \in A$ has at least as much neighbors in $C \cup D$ as a. Hence for each neighbor $x' \in C \cup D$ of a' either $(a', v)(x', u)$ or $(x', v)(a', u)$ is in S.

Altogether $(a, v), (a, u), (a', v)$ and (a', u) yield k grey cross edges between G_u and G_v that have another endvertex in $p_G^{-1}(C \cup D)$, where k is at least twice the number of neighbors of a in $C \cup D$. Thus we derive there are at least $2 \deg_G(a) \geq 2\lambda(G)$ cross edges in S.

We now prove the last assertion of the lemma. Suppose that the number of grey cross edges is exactly $2\lambda(G)$. First observe that $|A| \neq |B|$ implies that $2|B| < |A|+|B|$ in which case the number of grey cross edges is strictly greater than $2\lambda(G)$. Hence we may assume $|A| = |B|$. Let $a \in A \cup B$ be a vertex with the minimum number of neighbors in $C \cup D$. If there is a vertex $a' \in A \cup B$ with more neighbors in $C \cup D$ as a, then again we derive there are more than $2\lambda(G)$ grey cross edges. Hence all vertices from $A \cup B$ have the same number of neighbors in $C \cup D$. If $|A \cup B| \geq 4$ we find that next to $2 \deg_G(a)$ grey cross edges described in the previous paragraph, there are additional grey cross edges, namely $(a'', v)(x, u)$ or $(a'', u)(x, v)$, where $a'' \in A \cup B$ and $a'' \notin \{a, a'\}$ (where $x \in C \cup D$ is a neighbor of a''). Hence the last case is when A and B have only one vertex. Let $A = \{a\}$ and $B = \{b\}$ and denote

$$|N_G(a) \cap C| = \alpha, |N_G(a) \cap D| = \beta, |N_G(b) \cap C| = \gamma \text{ and } |N_G(b) \cap D| = \delta\,.$$

Since $\deg_G(a) = \deg_G(b)$ we find that $\alpha+\beta = \gamma+\delta$. If $\deg_G(a) > \lambda(G)$, then clearly the number of grey cross edges is greater than $2\lambda(G)$, a contradiction. So assume that $\deg_G(a) = \lambda(G)$. Furthermore there are no edges between C and D (for otherwise again there are strictly more than $2 \deg_G(a)$ grey cross edges). Hence vertices of C (and D) may be adjacent only to a and b. Hence $\alpha + \gamma \geq \lambda(G)$ and $\beta + \delta \geq \lambda(G)$. We get

$$\alpha + \gamma + \beta + \delta \geq 2\lambda(G).$$

On the left-hand side we have the sum of the number of neighbors of a from $C \cup D$ with the number of neighbors of b from $C \cup D$; on the right-hand side we have $\deg_G(a) + \deg_G(b)$. This readily implies that there must be the equality sign, and in addition, a and b are not adjacent. But a and b not adjacent implies there are at least $2\lambda(G) + 2$ grey cross edges between G_u and G_v, a contradiction. We infer that A and B are empty as desired. ■

3. Proof of Theorem 1

We first prove that

$$\lambda(G \boxtimes H) \leq \min\{\delta(G \boxtimes H), \lambda(G)(|V(H)| + 2|E(H)|), \lambda(H)(|V(G)| + 2|E(G)|)\}. \tag{1}$$

First, it is clear that $\lambda(G \boxtimes H) \leq \delta(G \boxtimes H)$. Let S' be any λ-set in G and let C_1' and C_2' be connected components of $G - S'$. Note that in $G \boxtimes H$ there are exactly $\lambda(G)(|V(H)| + 2|E(H)|)$ edges between $C_1' \times V(H)$ and $C_2' \times V(H)$. Hence $\lambda(G \boxtimes H) \leq \lambda(G)(|V(H)| + 2|E(H)|)$. Similarly we see that $\lambda(G \boxtimes H) \leq \lambda(H)(|V(G)| + 2|E(G)|)$. The inequality (1) follows.

Next we prove that

$$\lambda(G \boxtimes H) \geq \min\{\delta(G \boxtimes H), \lambda(G)(|V(H)| + 2|E(H)|), \lambda(H)(|V(G)| + 2|E(G)|)\}. \tag{2}$$

Let S be a λ-set in $G \boxtimes H$ and let C_1 and C_2 be connected components of $G \boxtimes H - S$. Recall that we call the vertices from C_1 black vertices and vertices from C_2 white vertices. We distinguish three cases (two of which are symmetric).

Case 1. No G-fiber lies entirely in one connected component of $G \boxtimes H - S$. In this case for every edge $uv \in E(H)$, $G_u \cap C_i$ and $G_v \cap C_i$ are nonempty for $i = 1, 2$. Therefore, by Lemma 3, for every edge $uv \in E(H)$ there are at least $2\lambda(G)$ grey cross edges (that is, edges with one endvertex in G_u and the other in G_v). As no G-fiber lies entirely in one connected component, we find that for each vertex $u \in V(H)$, there are at least $\lambda(G)$ grey edges in the subgraph induced by G_u. Summing up we infer that $|S| \geq \lambda(G)(|V(H)| + 2|E(H)|)$. Moreover, if the equality holds, then by Lemma 3, for every edge $uv \in E(H)$, $p_G(G_u \cap C_i) = p_G(G_v \cap C_i)$ for $i = 1, 2$. Since H is connected, this yields case (ii) of the theorem.

Case 2. No H-fiber lies entirely in one connected component of $G \boxtimes H - S$. By reversing the roles of G and H we infer by Case 1 that $|S| \geq \lambda(H)(|V(G)| + 2|E(G)|)$. Moreover, if the equality holds, case (iii) of the theorem occurs.

Case 3. A G-fiber and an H-fiber exist that both lie in one and the same connected component of $G \boxtimes H - S$. Without loss of generality we may assume, there is a G-fiber with only white vertices, and an H-fiber with only white vertices.

Set $\delta_1 = \delta(G)$ and $\delta_2 = \delta(H)$. Note that

$$\delta(G \boxtimes H) = \delta_1\delta_2 + \delta_1 + \delta_2.$$

We may assume without loss of generality that $\delta_1 \geq \delta_2$. Denote by W the projection $p_H(W')$, where W' consists of the G-fibers that have only white vertices. Since H is connected, there is a vertex $y \in V(H) \setminus W$ that is adjacent to some vertex in W. Since y is not in W there must be a black vertex in G_y. In fact, a black vertex (x, y) in G_y exists that is adjacent to a white vertex in G_y (this is because G_y does not contain only black vertices and because G is connected).

Let D be the set of neighbors of y that lie in W, and let C be the set of other neighbors of y, that is $C = N_H(y) \setminus D$. Set $|D| = d, |C| = c$, and note that $c+d \geq \delta_2$. Let K' be the set of black neighbors of (x, y) in G_y, and L' the set of white neighbors of (x, y) in G_y. Let $L = p_G(L')$ and $K = p_G(K')$ be the corresponding projections, and set $|L| = \ell, |K| = k$. Note that $k+\ell \geq \delta_1$. Also note that by the construction we have $d \geq 1, c \geq 0, \ell \geq 1$, and $k \geq 0$. See also Figure 3 where vertices whose color is not yet determined are grey.

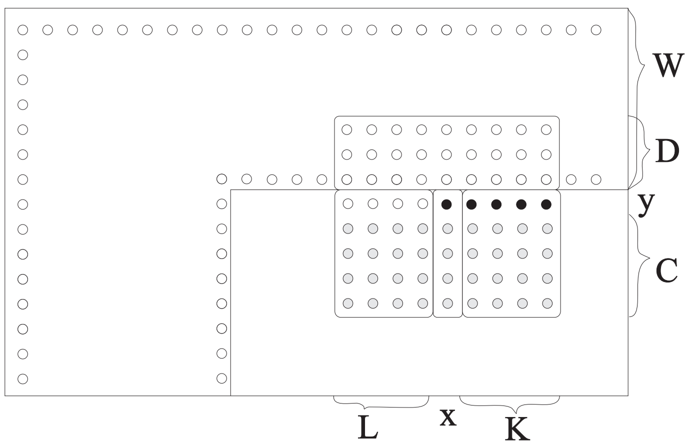

Figure 1. Case 3 in the proof of the theorem.

Note that in $V(G) \times D$ vertex (x, y) has $d(k + \ell + 1)$ neighbors that are all white, hence the respective edges are all grey. Also, every vertex from K' (these are black vertices in G_y) has in $V(G) \times D$ at least $d(\delta_1 + 1)$ neighbors. Summing this up for all vertices of K' we find there are at least $kd(\delta_1 + 1)$ grey edges. Next, (x, y) and its (white) neighbors from L' yield another ℓ grey edges. Consider now the vertices from $L \times C$ and note they are adjacent to (x, y). Let (u, v) be an arbitrary vertex from $L \times C$. If (u, v) is white, we find that the edge between (x, y) and (u, v) is grey. On the other hand, if (u, v) is black, then the edge between (u, y) and (u, v) is grey. In any case, each vertex of $L \times C$ is incident with at least one grey edge, which yields another $\ell \cdot c$ grey edges. Finally, the same argument can be applied for vertices from $\{x\} \times C$. They are all neighbors of (x, y) and also of $(u, y) \in L'$. Hence we get c grey edges that have an endvertex in $\{x\} \times C$.

Summing all up, we find the following inequality:

$$|S| \geq d(k + \ell + 1) + kd(\delta_1 + 1) + \ell c + \ell + c.$$

Modifying the right-hand side a little, and noting that

$$(\ell + kd) + (c + d) \geq \delta_1 + \delta_2$$

we infer

$$|S| \geq d(k + \ell) + kd\delta_1 + \ell c + \delta_1 + \delta_2. \tag{3}$$

Now, we distinguish two cases.

Case 3.1. $c \geq \delta_2$.
Then $\ell c \geq \ell\delta_2$, $kd\delta_1 \geq k\delta_2$ (since by assumption $\delta_1 \geq \delta_2$), and $d(k + \ell) > 0$. Combining this with (3) we infer

$$|S| > k\delta_2 + \ell\delta_2 + \delta_1 + \delta_2 = (k + \ell)\delta_2 + \delta_1 + \delta_2 \geq \delta_1\delta_2 + \delta_1 + \delta_2,$$

as desired.

Case 3.2. $c < \delta_2$.
Now $d(k + \ell) \geq d\delta_1$ and $kd\delta_1 \geq kd\delta_2 \geq kdc \geq kc$ (in fact $kd\delta_1 > kc$, unless $k = 0$). Hence

$$d(k + \ell) + kd\delta_1 + \ell c \geq \delta_1 d + kc + \ell c = \delta_1 d + (k + \ell)c \geq \delta_1(d + c) \geq \delta_1\delta_2.$$

Combining this and (3) we derive

$$|S| \geq \delta_1\delta_2 + \delta_1 + \delta_2.$$

In addition, we observe that $|S|$ is strictly greater than $\delta(G \boxtimes H)$ unless $k = 0$ (since in the first case $|S|$ is strictly greater than $\delta(G \boxtimes H)$).

Hence we may assume that $k = 0$, that is, (x, y) has no black neighbors in G_y. If there is some other black vertex $(u, v) \neq (x, y)$ in $G \boxtimes H$, then $v \neq y$ and we can choose (u, v) so that (u, v) has a white neighbor in G_v. This yields an additional grey edge in the G_v-fiber. But then $|S| > \deg_{G\boxtimes H}(x, y) \geq \delta(G \boxtimes H)$, a contradiction with S being a λ-set. We conclude that (x, y) is the only black vertex and so the case (i) of the theorem occurs. ∎

4. Concluding Remarks

Usually, when one takes two graphs G and H, the minimum for $\lambda(G \boxtimes H)$ will be achieved in $\delta(G \boxtimes H)$ (for instance, if G and H are paths or cycles). To see that also the other two cases of the theorem are applicable we present the following example.

Let G be obtained from the disjoint union of complete graphs K_p and K_r by adding a bridge (any edge between two vertices of the complete graphs). Let H be the complete graph K_m such that $m < p \leq r$. Then $\delta(G \boxtimes H) = pm - 1$ while $\lambda(G)(|V(H)| + 2|E(H)|) = m^2$ which is less than $pm - 1$ as soon as $m \geq 2$.

It might be an interesting problem to characterize strong products of graphs for which λ-sets have the structure from case (i) of the theorem.

The problem of finding a result similar to Theorem 1 for the edge-connectivity (and also vertex-connectivity) of the direct product of graphs seems to be very intriguing (we believe it is more difficult than the case of strong product).

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