Connections between Fairness Criteria and Efficiency for Allocating Indivisible Chores

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ABSTRACT

We study several fairness notions in allocating indivisible *chores* (i.e., items with non-positive values): envy-freeness and its relaxations. For allocations under each fairness criterion, we establish their approximation guarantees for other fairness criteria. Under the setting of additive cost functions, our results show strong connections between these fairness criteria and, at the same time, reveal intrinsic differences between goods allocation and chores allocation. Furthermore, we investigate the efficiency loss under these fairness constraints and establish their *prices of fairness*.

KEYWORDS

Fair Division; Indivisible Chores; Price of Fairness

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1 INTRODUCTION

Fair division is a central matter of concern in economics, multiagent systems, and artificial intelligence [6, 14, 16]. Over the years, there emerges a tremendous demand for fair division when a set of indivisible resources, such as classrooms, tasks, and properties, are divided among a group of n agents. This field has attracted the attention of researchers and most results are established when resources are considered as goods that bring positive utility to agents. However, in real-life division problems, the resources to be allocated can also be chores which, instead of positive utility, bring non-positive utility or cost to agents. For example, one might need to assign tasks among workers, teaching load among teachers, sharing noxious facilities among communities, and so on. Compared to goods, fairly dividing chores is relatively under-developed. At first glance, dividing chores is similar to dividing goods. However, in general, chores allocation is not covered by goods allocation and results established on goods do not necessarily hold on chores. Studies in [12, 13, 17] and [26, 27] have already pointed out this difference in the context of envy-freeness and equitability, respectively. As an example [26], when allocating goods a *leximin*[1] allocation is Pareto optimal and *equitable up to any item*[2], however, a leximin solution does not guarantee equitability up to any item in chores allocation.

Among the variety of fairness notions introduced in the literature, *envy-freeness* (EF) is one of the most compelling ones, which has drawn research attention over the past few decades [15, 19, 25]. In an envy-free allocation, no agent envies another agent. Unfortunately, the existence of an envy-free allocation cannot be guaranteed in general when the items to be assigned are indivisible. A canonical example is that one needs to assign one chore to two agents and the chore has a positive cost for either agent. Clearly, the agent who receives the chore will envy the other. In addition, deciding the existence of an EF allocation is computationally intractable, even for two agents with identical preference [32]. Given this predicament, recent studies mainly devote to relaxations of envy-freeness. One direct relaxation is known as *envy-free up to one item* (EF1) [18, 32]. In an EF1 allocation, one agent may be jealous of another, but by removing one chore from the bundle of the envious agent, envy can be eliminated. A similar but stricter notion is *envy-free up to any item* (EFX) [21]. In such an allocation, envy can be eliminated by removing any positive-cost chore from the envious agent's bundle. Another fairness notion, *maximin share* (MMS) [3, 18], generalizes the idea of "cut-and-choose" protocol in cake cutting. The maximin share is obtained by minimizing the maximum cost of a bundle of an allocation over all allocations. The last fairness notion we consider is called *pairwise maximin share* (PMMS) [21], which is similar to maximin share but different from MMS in that each agent partitions the combined bundle of himself and any other agent into two bundles and then receives the one with the larger cost.

The existing research on envy-freeness and its relaxations concentrates on algorithmic features of fairness criteria, such as their existence and (approximation) algorithms for finding them. Relatively little research studies the connections between these fairness criteria themselves, or the trade-off between these fairness criteria and the system efficiency, known as *the price of fairness*. When allocating goods, Amanatidis et al. [2] compare the four aforementioned relaxations of envy-freeness and provides results on the approximation guarantee of one to another. However, these connections are unclear in allocating chores. On the price of fairness, Bei et al. [9] study allocating indivisible goods and focuses on the notions for which corresponding allocations are guaranteed to exist, such as EF1, maximin Nash welfare[3], and leximin. Caragiannis et al.

[1] A leximin solution selects the allocation that maximizes the utility of the least well-off agent, subject to maximizing the utility of the second least, and so on.

[2] Equitability requires that any pair of agents are equally happy with their bundles. In equitability up to any item allocations, the violation of equitability can be eliminated by removing any single item from the happier (in goods allocation)/ less happy agent (in chores allocation).

[3] Nash welfare is the product of agents' utilities.

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[20] study the price of fairness for both chores and goods, and focuses on the classical fairness notions, namely, EF, proportionality[4] and equitability. When allocating chores, it provides a tight upper bound for the price of proportionality and also shows that the price of both envy-freeness and equitability are infinite (although such an allocation may not exist at all). However, in allocating chores, the price of fairness is still unknown for any of the aforementioned four relaxations of envy-freeness.

In this paper, we fill these gaps by investigating the four relaxations of envy-freeness on two aspects. On the one hand, we study the connections between these criteria and, in particular, we consider the following questions: *Does one fairness criterion implies another? To what extent can one criterion guarantee for another?* On the other hand, we study the trade-off between fairness and *efficiency* (or *social cost* defined as the sum of costs of the individual agents). Specifically, for each fairness criterion, we investigate its *price of fairness*, which is defined as the supremum ratio of the minimum social cost of a fair allocation to the minimum social cost of any allocation.

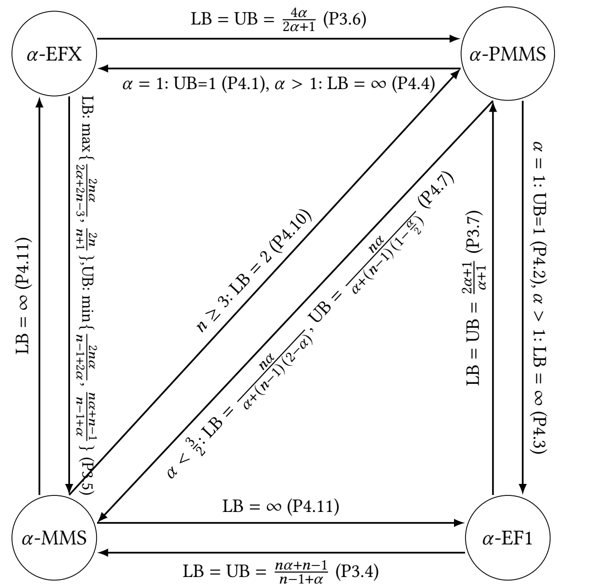
1.1 Main Results

On the connections between fairness criteria, we summarize our main results in Figure 1 on the approximation guarantee of one fairness criterion for another when the cost functions are additive, where α-Z (formally defined in Section 2) refers to α-approximation for fairness of notion Z. While some of our results show similarity to those in goods allocation [2], others also reveal the difference between allocating goods and chores.

After comparing each pair of fairness notions, we compare the efficiency of fair allocations with the optimal one. To quantify the efficiency loss, we apply the idea of the price of fairness and our results are summarized in Table 1.

Table 1: Prices of fairness, where P$x.y$ points to Proposition $x.y$

	EFX	PMMS	EF1	2-MMS	1.5-PMMS
$n=2$	2	2	$\frac{5}{4}$	1	$\frac{7}{6}$
	(P5.4)	(P5.4)	(P5.1)	(P5.2)	(P5.3)
$n\geq 3$	∞	∞	∞	$\Theta(n)$	∞
	(P5.5)	(P5.5)	(P5.5)	(P5.8)	(P5.6)



Note: LB and UB stand for lower and upper bound, respectively. For example, the directed edge from α-EFX to α-PMMS with label LB $=$ UB $= \frac{4\alpha}{2\alpha+1}$ means that α-EFX implies $\frac{4\alpha}{2\alpha+1}$-PMMS, and this result is tight. P$x.y$ points to Proposition $x.y$

Figure 1: Connections between fairness criteria

1.2 Related Works

The fair division problem has been studied for both indivisible goods [11, 21, 32] and indivisible chores [5, 7, 27]. Among various fairness notions, a prominent one is EF proposed in Foley [25]. But an EF allocation may not exist and even worse, checking the existence of an EF allocation is NP-complete [6]. For the relaxations of envy-freeness, Lipton et al. [32] originate the notion of EF1 and provides an efficient algorithm for EF1 allocations of goods when agents have monotone utility functions. For allocating chores, EF1 is achievable by allocating chores in a round-robin fashion if agents have additive cost functions [4]. Another fairness notion that has been a subject of much interest in the last few years is MMS, proposed by Budish [18]. However, existence of an MMS allocation is not guaranteed either for goods [31] or for chores [7], even with additive functions. Consequently, more efforts are on approximation of MMS, with [3, 28, 29] on goods allocation and [7, 30] on chores allocation. The notions of EFX and PMMS are introduced by Caragiannis et al. [21]. They consider goods allocation and establish that a PMMS allocation is also EFX when the valuation functions are additive. Beyond the simple case of $n = 2$, the existence of an EFX allocation has not been settled in general. However, significant results have been achieved for some special cases. When $n = 3$, the existence of an EFX allocation of goods is proved in Chaudhury et al. [22]. Based on a modified version of leximin solutions, Plaut and Roughgarden [33] show that an EFX allocation is guaranteed to exist when all agents have identical valuations. The work most related to ours is Amanatidis et al. [2], which is on goods allocation, and provides connections between the four EF relaxations.

As for the price of fairness, Caragiannis et al. [20] show that, in the case of *divisible* goods, the price of proportionality is $\Theta(\sqrt{n})$ and the price of equitability is $\Theta(n)$. Bertsimas et al. [10] extend the study to other fairness notions, *maximin*[5] fairness and proportional fairness, and provides a tight bound on the price of fairness for a broad family of problems. Bei et al. [9] focus on indivisible goods and concentrates on the fairness notions that are guaranteed to exist. The authors present an asymptotically tight upper bound

[4] An allocation of goods (resp. chores) is proportional if the value (resp. cost) of every agent's bundle is at least (resp. at most) one n-th fraction of his value (resp. cost) for all items.

[5] It maximizes the lowest utility level among all the agents.

of $\Theta(n)$ on the price of maximum Nash welfare [23], maximum egalitarian welfare [16] and leximin. They also consider the price of EF1 but leave a gap between the upper bound $O(n)$ and lower bound $\Omega(\sqrt{n})$. This gap is later closed by Barman et al. [8] with the results that, for both EF1 and $\frac{1}{2}$-MMS, the price of fairness is $O(\sqrt{n})$. In addition, the price of fairness has been studied in other topics of multi-agent systems, such as machine scheduling [1] and kidney exchange [24].

2 PRELIMINARIES

In a fair division problem on indivisible chores, we have a set $N = \{1, 2, \ldots, n\}$ of n agents and a set $E = \{e_1, \ldots, e_m\}$ of m indivisible chores. As chores are the items with non-positive values, each agent $i \in N$ is associated with a cost function $c_i : 2^E \to R_{\geq 0}$, which maps any subsets of E into a non-negative real number. In this paper, we assume $c_i(\emptyset) = 0$ and c_i is monotone, that is, $c_i(S) \leq c_i(T)$ for any $S \subseteq T \subseteq E$. We say a (set) function $c(\cdot)$ is *additive* if $c(S) = \sum_{e \in S} c(e)$ for any $S \subseteq E$. In the remainder of this paper, we assume all cost functions are additive. For simplicity, instead of $c_i(\{e_j\})$, we use $c_i(e_j)$ to represent the cost of chore e_j for agent i.

An *allocation* $\mathbf{A} := (A_1, \ldots, A_n)$ is an n-partition of E among agents in N, i.e., $A_i \cap A_j = \emptyset$ for any $i \neq j$ and $\cup_{i \in N} A_i = E$. Each subset $S \subseteq E$ also refers to a *bundle* of chores. For any bundle S and $k \in \mathbb{N}^+$, we denote by $\Pi_k(S)$ the set of all k-partition of S, and $|S|$ the number of chores in S.

2.1 Fairness Criteria

We study envy-freeness and its relaxations and are concerned with both exact and approximate versions of these fairness notions.

Definition 2.1. For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-EF if for any $i, j \in N, c_i(A_i) \leq \alpha \cdot c_i(A_j)$. In particular, 1-EF is simply called EF.

Definition 2.2. For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-EF1 if for any $i, j \in N$, there exists $e \in A_i$ such that $c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j)$. In particular, 1-EF1 is simply called EF1.

Definition 2.3. For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-EFX if for any $i, j \in N, c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j)$ for any $e \in A_i$ with $c_i(e) > 0$. In particular, 1-EFX is simply called EFX.

Clearly, EFX[6] is stricter than EF1. Next, we formally introduce the notion of maximin share. For any $k \in [n] = \{1, \ldots, n\}$ and bundle $S \subseteq E$, the maximin share of agent i on S among k agents is

$$\mathrm{MMS}_i(k, S) = \min_{A \in \Pi_k(S)} \max_{j \in [k]} c_i(A_j).$$

We are interested in the allocation in which each agent receives cost no more than his maximin share.

Definition 2.4. For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-MMS if for any $i \in N, c_i(A_i) \leq \alpha \cdot \mathrm{MMS}_i(n, E)$. In particular, 1-MMS is called MMS.

[6]Note Plaut and Roughgarden [33] consider a stronger version of EFX by dropping the condition $c_i(e) > 0$. In this paper, all results about EFX, except Propositions 4.1 and 4.6, still hold under the stronger version.

Definition 2.5. For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-PMMS if for any $i, j \in N$,

$$c_i(A_i) \leq \alpha \cdot \min_{\mathbf{B} \in \Pi_2(A_i \cup A_j)} \max\{c_i(B_1), c_i(B_2)\}.$$

In particular, 1-PMMS is called PMMS.

Note that the right-hand side of the above inequality is equivalent to $\alpha \cdot \mathrm{MMS}_i(2, A_i \cup A_j)$.

Example 2.6. Let us consider an example with three agents and a set $E = \{e_1, \ldots, e_7\}$ of seven chores. The cost functions of agents are shown as follows.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Agent 1	2	3	3	0	4	2	1
Agent 2	3	1	3	2	5	0	5
Agent 3	1	5	10	2	3	1	3

It is not hard to verify that $\mathrm{MMS}_1(3, E) = 5, \mathrm{MMS}_2(3, E) = 7, \mathrm{MMS}_3(3, E) = 10$. For instance, agent 2 can partition E into three bundles: $\{e_1, e_3\}, \{e_2, e_7\}, \{e_4, e_5, e_6\}$, so that the maximum cost of any single bundle for her is 7. Moreover, there is no other partitions that can guarantee a better worst-case cost.

Now examine allocation $\mathbf{A} = (A_1, A_2, A_3)$ with $A_1 = \{e_1, e_4, e_7\}$, $A_2 = \{e_2, e_3, e_6\}$ and $A_3 = \{e_5\}$. It is not hard to see that $\mathbf{A}$ is an EF allocation, and accordingly, it is also an EFX, EF1, MMS and PMMS allocation. For another allocation $\mathbf{B}$ with $B_1 = \{e_1, e_5, e_7\}, B_2 = \{e_2, e_4, e_6\}, B_3 = \{e_3\}$, agent 1 would still envy agent 2 even if chore e_7 is eliminated from her bundle, and hence, allocation $\mathbf{B}$ is neither exact EF nor EFX. One can verify that $\mathbf{B}$ is indeed $\frac{7}{3}$-EF and 2-EFX. Moreover, $\mathbf{B}$ is an EF1 allocation because agent 1 would not envy others if chore e_5 is eliminated from her bundle and agent 3 would not envy others if chore e_3 is eliminated from her bundle. As for the approximation guarantee on the notions of MMS and PMMS, it is not hard to verify that allocation $\mathbf{B}$ is $\frac{7}{5}$-MMS and $\frac{7}{5}$-PMMS.

2.2 Price of Fairness

Let $I = \langle N, E, (c_i)_{i \in N} \rangle$ be an instance of the problem for allocating indivisible chores and let $\mathcal{I}$ be the set of all such instances. The *social cost* of an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is defined as $\mathrm{SC}(\mathbf{A}) = \sum_{i \in N} c_i(A_i)$. The optimal social cost for an instance I, denoted by $\mathrm{OPT}(I)$, is the minimum social cost over all allocations for this instance. Following previous work [9, 20], when study the price of fairness, we assume that agents cost functions are normalized to one, i.e., $c_i(E) = 1$ for all $i \in N$.

The *price of fairness* is the supremum ratio over all instances between the social cost of the "best" fair allocation and the optimal social cost, where "best" refers to the one with the minimum cost. Since we consider several fairness criteria, let F be any given fairness criterion and define by $F(I)$ as the set (possibly empty) of all allocations for instance I that satisfy fairness criterion F.

Definition 2.7. For any given fairness property F, the price of fairness with respect to F is defined as

$$\mathrm{PoF} = \sup_{I \in \mathcal{I}} \min_{\mathbf{A} \in F(I)} \frac{\mathrm{SC}(\mathbf{A})}{\mathrm{OPT}(I)},$$

where PoF is equal to $+\infty$ if $F(I) = \emptyset$.

2.3 Simple Facts

We begin with some initial results, which reveal some intrinsic difference in allocating goods and allocating chores as far as approximation guarantee is concerned. Due to space constraint, proofs of these results are omitted, which can be found in [34]. First, we state a simple lemma concerning lower bounds of the maximin share.

Lemma 2.8. *For any agent $i \in N$ and bundle $S \subseteq E$,*

- $\mathrm{MMS}_i(k, S) \geq \frac{1}{k} c_i(S), \forall k \in [n]$;
- $\mathrm{MMS}_i(k, S) \geq c_i(e), \forall e \in S, \forall k \in [n]$.

Based on the lower bounds in Lemma 2.8, we provide a trivial approximation guarantee for PMMS and MMS.

Lemma 2.9. *Any allocation is 2-*PMMS *and n-*MMS*.*

As can be seen from the proof of Lemma 2.9, in allocating chores, if one assigns all chores to one agent, then the allocation still has a bounded approximation for PMMS and MMS. However, when allocating goods, if an agent receives nothing but his maximin share is positive, then clearly the corresponding allocation has an infinite approximation guarantee for PMMS and MMS.

3 PERFORMANCE BOUNDS ON EF, EFX, AND EF1

Let us start with EF. According to the definitions, for any $\alpha \geq 1$, α-EF is stronger than α-EFX and α-EF1. The following proposition presents an approximation guarantee of α-EF for MMS and PMMS.

Proposition 3.1. *For any $\alpha \geq 1$, an α-*EF *allocation is also $\frac{n\alpha}{n-1+\alpha}$-*MMS*, and this result is tight.*

Proposition 3.2. *For any $\alpha \geq 1$, an α-*EF *allocation is also $\frac{2\alpha}{1+\alpha}$-*PMMS*, and this result is tight.*

Proposition 3.2 indicates that the approximation guarantee of α-EF for PMMS is independent of the number of agents. However, according to Proposition 3.1, its approximation guarantee for MMS is affected by the number of agents. Moreover, this guarantee ratio converges to α as n goes to infinity.

We remark that none of EFX, EF1, PMMS and MMS has a bounded guarantee for EF. We show this by a simple example. Consider an instance of two agents and one chore, and the chore has a positive cost for both agents. Assigning the chore to an arbitrary agent results in an allocation that satisfies EFX, EF1, PMMS and MMS, simultaneously. However, since one agent has a positive cost on his own bundle and zero cost on other agents' bundle, such an allocation has an infinite approximation guarantee for EF.

Next, we consider approximation of EFX and EF1.

Proposition 3.3. *An α-*EFX *allocation is α-*EF1 *for any $\alpha \geq 1$. On the other hand, an* EF1 *allocation is not β-*EFX *for any $\beta \geq 1$.*

Next, we consider the approximation guarantee of EF1 for MMS. In allocating goods, Amanatidis et al. [2] present a tight result that an α-EF1 allocation is $O(n)$-MMS. In contrast, in allocating chores, α-EF1 can have a much better guarantee for MMS.

Proposition 3.4. *For any $\alpha \geq 1$ and $n \geq 2$, an α-*EF1 *allocation is also $\frac{n\alpha+n-1}{n-1+\alpha}$-*MMS*, and this result is tight.*

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-EF1 allocation and we focus on agent i. If $A_i = \emptyset$ or A_i only contains chores with zero cost for agent i, then $c_i(A_i) = 0$ which would violate neither α-EF1 nor MMS. Thus, without loss of generality, we assume $A_i \neq \emptyset$ and A_i contains chores with strictly positive cost for agent i. Let $\bar{e}$ be the chore with largest cost for agent i in bundle A_i, i.e., $\bar{e} \in \arg\max_{e \in A_i} c_i(e)$.

By the definition of α-EF1, for any $j \in N \setminus \{i\}$, $c_i(A_i \setminus \{\bar{e}\}) \leq \alpha \cdot c_i(A_j)$ holds. Then, by summing up j over $N \setminus \{i\}$ and adding a term $\alpha c_i(A_i)$ on both sides, the following holds,

$$\alpha \cdot \sum_{j \in N} c_i(A_j) \geq (n - 1 + \alpha) c_i(A_i) - (n-1) c_i(\bar{e}). \tag{1}$$

From Lemma 2.8, we have $\mathrm{MMS}_i(n, E) \geq \max\{\frac{1}{n} c_i(E), c_i(\bar{e})\}$, and by additivity, it holds that

$$n\alpha \mathrm{MMS}_i(n, E) \geq (n - 1 + \alpha) c_i(A_i) - (n-1) \mathrm{MMS}_i(n, E). \tag{2}$$

Inequality (2) is equivalent to $\frac{c_i(A_i)}{\mathrm{MMS}_i(n,M)} \leq \frac{n\alpha+n-1}{n-1+\alpha}$, as required.

As for tightness, consider an instance with n agents and a set $E = \{e_1, \dots, e_{n^2-n+1}\}$ of $n^2 - n + 1$ chores. Agents have identical cost profile. The cost function of agent 1 is as follow:

$$c_1(e_j) = \begin{cases} \alpha + n - 1, & j = 1, \\ \alpha, & 2 \leq j \leq n, \\ 1, & j \geq n + 1. \end{cases}$$

Now, consider an allocation $\mathbf{B} = \{B_1, \dots, B_n\}$ with $B_1 = \{e_1, \dots, e_n\}$ and $B_j = \{e_{n+(n-1)(j-2)+1}, \dots, e_{n+(n-1)(j-1)}\}$ for any $j \geq 2$. Since agents have identical cost profile, for any agent i and bundle B_j with $j \geq 2$, $c_i(B_j) = c_1(B_j) = n - 1$, smaller than the cost of bundle B_1. Accordingly, except for agent 1, no one else will violate the condition of α-EF1 and MMS. As for agent 1, since $c_1(B_1 \setminus \{e_1\}) = (n-1)\alpha = \alpha c_1(B_j), \forall j \geq 2$, then we can claim that allocation $\mathbf{B}$ is α-EF1. To calculate $\mathrm{MMS}_1(n, E)$, consider an allocation $\mathbf{T} = (T_1, \dots, T_n)$ with $T_1 = \{e_1\}$ and $T_j = \{B_j \cup \{e_j\}\}$ for any $2 \leq j \leq n$. It is not hard to verify that $c_1(T_j) = \alpha + n - 1$ for any $j \in N$. Therefore, we have $\mathrm{MMS}_1(n, E) = \alpha + n - 1$ implying the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(n,E)} = \frac{n\alpha+n-1}{n-1+\alpha}$, completing the proof. □

We now study α-EFX in terms of its approximation guarantee for MMS and provide upper and lower bounds for general $\alpha \geq 1$ or $n \geq 2$.

Proposition 3.5. *When agents have additive cost functions, for any $\alpha \geq 1$ and $n \geq 2$, an α-*EFX *allocation is $\min\left\{\frac{2n\alpha}{n-1+2\alpha}, \frac{n\alpha+n-1}{n-1+\alpha}\right\}$-*MMS*, while it is not β-*MMS *for any $\beta < \max\left\{\frac{2n\alpha}{2\alpha+2n-3}, \frac{2n}{n+1}\right\}$.*

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-EFX allocation with $\alpha \geq 1$ and we focus on agent i. The upper bound $\frac{n\alpha+n-1}{n-1+\alpha}$ directly follows from Proposition 3.3 and 3.4. In what follows, we prove the upper bound $\frac{2n\alpha}{n-1+2\alpha}$. If $A_i = \emptyset$ or A_i only contains chores with zero cost for agent i, then $c_i(A_i) = 0$ which would violate the condition of neither MMS nor α-EFX. Thus, without loss of generality, we assume that $A_i \neq \emptyset$ and meanwhile contains chores with a strictly positive cost for agent i. Let e^* be the chore in bundle A_i having the minimum cost for agent i, i.e., $e^* \in \arg\min_{e \in A_i} c_i(e)$. Next, we divide the proof into two cases.

Case 1: $|A_i| = 1$. Then e^* is the unique element in A_i, and thus $c_i(A_i) = c_i(e^*)$. By the second point of Lemma 2.8, $c_i(e^*) \leq \text{MMS}_i(n, E)$ holds, and thus, $c_i(A_i) \leq \text{MMS}_i(n, E)$.

Case 2: $|A_i| \geq 2$. By the definition of α-EFX, for any agent $j \in N \setminus \{i\}$, $c_i(A_i \setminus \{e^*\}) \leq \alpha \cdot c_i(A_j)$. Since $e^* \in \arg\min_{e \in A_i} c_i(e)$ and $|A_i| \geq 2$, we have $c_i(e^*) \leq \frac{1}{2} c_i(A_i)$. Then, the following holds,

$$\alpha c_i(A_j) \geq c_i(A_i) - c_i(e^*) \geq \frac{1}{2} c_i(A_i), \qquad \forall j \in N \setminus \{i\}. \qquad (3)$$

By summing up j over $N \setminus \{i\}$ and adding a term $\alpha c_i(A_i)$ on both sides of inequality (3), the following holds

$$\alpha c_i(E) = \alpha \sum_{j \in N \setminus \{i\}} c_i(A_j) + \alpha c_i(A_i) \geq \frac{n - 1 + 2\alpha}{2} c_i(A_i). \qquad (4)$$

On the other hand, from the first point of Lemma 2.8, we know $\text{MMS}_i(n, E) \geq \frac{1}{n} c_i(E)$, which combines inequality (4) yielding the ratio

$$\frac{c_i(A_i)}{\text{MMS}_i(n, M)} \leq \frac{2n\alpha}{n - 1 + 2\alpha}.$$

Regarding the lower bound $\frac{2n}{n+1}$, consider an instance with n agents and a set $E = \{e_1, e_2, ..., e_{2n}\}$ of $2n$ chores. Agents have identical cost profile. The cost function of agent 1 is $c_1(e_j) = \lceil \frac{j}{2} \rceil$ for any $j \geq 1$. It is easy to see $\text{MMS}_i(n, E) = n + 1$ for any agent i. Then, consider an allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_1 = \{e_{2n-1}, e_{2n}\}$ and $B_i = (e_{i-1}, e_{2n-i})$ for any $i \geq 2$. Since agents have identical profile, for any agent i and bundle B_j with $j \geq 2$, we have $c_i(B_j) = c_1(B_j) = n$. Thus, except for agent 1, no one else will violate the condition of MMS and EFX. As for agent 1, envy can be eliminated by removing any single chore since $c_1(B_1 \setminus \{e_{2n}\}) = c_1(B_1 \setminus \{e_{2n-1}\}) = n$. Hence, the allocation $\mathbf{B}$ is EFX and its approximation guarantee on MMS equals to $\frac{c_1(B_1)}{\text{MMS}_1(n,E)} = \frac{2n}{n+1}$, as required.

Next, for lower bound $\frac{2n\alpha}{2\alpha+2n-3}$, let us consider an instance with n agents and a set $E = \{e_1, ..., e_{2n^2-2n}\}$ of $2n^2 - 2n$ chores. We focus on agent 1 and his cost function is $c_1(e_j) = 2\alpha$ for $j \leq n$ and $c_1(e_j) = 1$ for $j \geq n + 1$. Now, consider an allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_1 = \{e_1, ..., e_n\}, B_2 = \{e_{n+1}, ..., e_{3n-2}\}$ and $B_j = \{e_{3n-1+(j-3)(2n-1)}, ..., e_{3n-2+(j-2)(2n-1)}\}$ for any $j \geq 3$. Accordingly, bundle B_2 contains $2n - 2$ chores and B_j contains $2n - 1$ chores for any $j \geq 3$. For $i \geq 2$, every agent i has cost $0 < \delta < \epsilon$ on each single chore in bundle B_i with δ arbitrarily small, while his cost on other chores are one. Consequently, except for agent 1, no one else will violate the condition of MMS and α-EFX. As for agent 1, his cost on B_2 is the smallest over all bundles and $c_1(B_1 \setminus \{e_1\}) = 2\alpha(n - 1) = \alpha c_1(B_2)$, as a result, the allocation $\mathbf{B}$ is α-EFX. For $\text{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\text{MMS}_1(n, E) = 2\alpha + 2n - 3$ implying the ratio $\frac{c_1(B_1)}{\text{MMS}_1(n,E)} = \frac{2n\alpha}{2\alpha+2n-3}$, completing the proof. □

The upper bound in Proposition 3.5 is almost tight since $\frac{n\alpha+n-1}{n-1+\alpha} - \frac{2n\alpha}{2\alpha+2n-3} < \frac{n-1}{n-1+\alpha} < 1$. In addition, we highlight that the upper and lower bounds provided in Proposition 3.5 are tight in two interesting cases: (i) $\alpha = 1$ and (ii) $n = 2$.

On the approximation of EFX and EF1 for PMMS, we have the following propositions.

Proposition 3.6. *For any $\alpha \geq 1$, an α-*EFX *allocation is also $\frac{4\alpha}{2\alpha+1}$-*PMMS*, and this guarantee is tight.*

Proposition 3.7. *For any $\alpha \geq 1$, an α-*EF1 *allocation is also $\frac{2\alpha+1}{\alpha+1}$-*PMMS*, and this guarantee is tight.*

In addition to the approximation guarantee for PMMS, Proposition 3.7 also has a direct implication in approximating PMMS algorithmically. It is known that an EF1 allocation can be found efficiently by allocating chores in a *round-robin* fashion — agents in turn pick their most preferred chores from the remaining [4]. Therefore, Proposition 3.7 with $\alpha = 1$ leads to the following corollary, which is the only algorithmic result for PMMS (in chores allocation), to the best of our knowledge.

Corollary 3.8. *The round-robin algorithm outputs a $\frac{3}{2}$-*PMMS *allocation in polynomial time.*

4 PERFORMANCE BOUNDS ON PMMS AND MMS

Note that PMMS implies EFX in goods allocation according to Caragiannis et al. [21]. This implication also holds in allocating chores as stated in our proposition below.

Proposition 4.1. *A* PMMS *allocation is also* EFX.

Since EFX implies EF1, Proposition 4.1 directly leads to the following corollary.

Corollary 4.2. *A* PMMS *allocation is also* EF1.

For approximate version of PMMS, when allocating goods it is shown in Amanatidis et al. [2] that for any α, α-PMMS can imply $\frac{\alpha}{2-\alpha}$-EF1. However, in the case of chores, our results indicate that α-PMMS has no bounded guarantee for EF1.

Proposition 4.3. *For $n \geq 2$, an α-*PMMS *allocation with $\alpha > 1$ is not necessarily β-*EF1 *for any $\beta \geq 1$.*

Proof. First note according to Lemma 2.9 that we can assume without loss of generality that $1 < \alpha < 2$. Consider an instance with n agents and $n + 1$ chores $e_1 \ldots, e_{n+1}$. Agents have identical cost profile. For any agent i, the cost function is as follow: $c_i(e_1) = \frac{1}{\alpha-1}$, $c_i(e_2) = 1$ and $c_i(e_j) = \epsilon, \forall j \geq 3$ where ϵ takes any arbitrarily small positive value. Then, consider an allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_1 = \{e_1, e_2\}$ and $B_j = \{e_{j+1}\}, \forall j \geq 2$. Consequently, except for agent 1, no one else will violate the condition of EF1 and α-PMMS. As for agent 1, notice that $\frac{1}{\alpha-1} > 1+\epsilon$ and thus, for any $j \geq 2$, the combined bundle $B_1 \cup B_j$ admits $\text{MMS}_1(2, B_1 \cup B_j) = \frac{1}{\alpha-1}$ that implies $\frac{c_1(B_1)}{\text{MMS}_1(2,B_1 \cup B_j)} = \alpha$. Thus, allocation $\mathbf{B}$ is α-PMMS. For the guarantee on EF1, as $c_1(B_j) = \epsilon$ for any $j \geq 2$, then removing the chore with the largest cost from B_2 still yields the ratio $\frac{c_1(B_1 \setminus \{e_1\})}{c_1(B_j)} = \frac{1}{\epsilon} \to \infty$ as $\epsilon \to 0$, completing the proof. □

Since for any $\alpha \geq 1$, α-EFX is stricter than α-EF1, the impossibility result on EF1 in Proposition 4.3 is also true for EFX.

Proposition 4.4. *For $n \geq 2$, an α-*PMMS *allocation with $\alpha > 1$ is not necessarily a β-*EFX *allocation for any $\beta \geq 1$.*

We now study the approximation guarantee of PMMS for MMS. Since these two notions coincide when there are only two agents, we assume there are at least three agents. We first provide a tight bound for $n = 3$ and then give an almost tight bound for general n.

Proposition 4.5. *For* $n = 3$, *a* PMMS *allocation is also* $\frac{4}{3}$-MMS, *and moreover, this bound is tight.*

For general n, we use the connections between PMMS, EFX and MMS to find the approximation guarantee of PMMS for MMS. According to Proposition 4.1, a PMMS allocation is also EFX, and by Proposition 3.5, EFX implies $\frac{2n}{n+1}$-MMS. As a result, we can claim that PMMS also implies $\frac{2n}{n+1}$-MMS. With the following proposition we show that this guarantee is almost tight.

Proposition 4.6. *For* $n \geq 4$, *a* PMMS *allocation is* $\frac{2n}{n+1}$-MMS *but not necessarily* $(\frac{2n+2}{n+3} - \epsilon)$-MMS *for any* $\epsilon > 0$.

Next, we investigate the approximation guarantee of approximate PMMS for MMS. Let us start with an example of six chores $E = \{e_1, \ldots, e_6\}$ and three agents. We focus on agent 1 and the cost function of agent 1 is $c_1(e_j) = 1$ for $j = 1, 2, 3$ and $c_1(e_j) = 0$ for $j = 4, 5, 6$, thus clearly, $\text{MMS}_1(3, E) = 1$. Consider an allocation $\mathbf{A} = (A_1, A_2, A_3)$ with $A_1 = \{e_1, e_2, e_3\}$. It is not hard to verify that allocation $\mathbf{A}$ is a $\frac{3}{2}$-PMMS allocation and also a 3-MMS allocation. Combining the result in Lemma 2.9, we observe that allocation $\mathbf{A}$ only has a trivial guarantee on the notion of MMS. Motivated by this example, we focus on α-PMMS allocations with $\alpha < \frac{3}{2}$.

Proposition 4.7. *For any* $n \geq 3$ *and* $1 < \alpha < \frac{3}{2}$, *an* α-PMMS *allocation is* $\frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$-MMS, *but not necessarily* $(\frac{n\alpha}{\alpha+(n-1)(2-\alpha)} - \epsilon)$-MMS *for any* $\epsilon > 0$.

Before we can prove the above proposition, we need the following two lemmas.

Lemma 4.8. *For any* $i \in N$ *and bundle* $S \subseteq E$, *suppose* $\text{MMS}_i(2, S)$ *is defined by a 2-partition* $\mathbf{T} = (T_1, T_2)$ *with* $c_i(T_1) = \text{MMS}_i(2, S)$. *If the number of chores in* T_1 *is at least two, then* $\frac{c_i(S)}{\text{MMS}_i(2,S)} \geq \frac{3}{2}$.

Lemma 4.9. *For any* $i \in N$ *and bundles* $S_1, S_2 \subseteq E$, *if* $\text{MMS}_i(2, S_1 \cup S_2) > \text{MMS}_i(2, S_1)$, *then* $\text{MMS}_i(2, S_1 \cup S_2) \leq \frac{1}{2}c_i(S_1) + c_i(S_2)$.

Proof of Proposition 4.7. We first prove the upper bound. Let $\mathbf{A} = (A_1, ..., A_n)$ be an α-PMMS allocation and we focus our analysis on agent i. Let $\alpha^{(i)} = \max_{j \neq i} \frac{c_i(A_i)}{\text{MMS}_i(2, A_i \cup A_j)}$ and $j^{(i)}$ be the index such that $\text{MMS}_i(2, A_i \cup A_{j^{(i)}}) \leq \text{MMS}_i(2, A_i \cup A_j)$ for any $j \in N$ (tie breaks arbitrarily). By these constructions, clearly, $\alpha = \max_{i \in N} \alpha^{(i)}$ and $c_i(A_i) = \alpha^{(i)} \cdot \text{MMS}_i(2, A_i \cup A_{j^{(i)}})$. Then, we split our proof into two different cases.

Case 1: $\exists j \neq i$ such that $\text{MMS}_i(2, A_i \cup A_j) = \text{MMS}_i(2, A_i)$. Then $\alpha^{(i)} = \frac{c_i(A_i)}{\text{MMS}_i(2,A_i)}$ holds. Suppose $\text{MMS}_i(2, A_i)$ is defined by the 2-partition (T_1, T_2) with $c_i(T_1) = \text{MMS}_i(2, A_i)$. If $|T_1| \geq 2$, by Lemma 4.8, we have $\alpha^{(i)} = \frac{c_i(A_i)}{\text{MMS}_i(2,A_i)} \geq \frac{3}{2}$, contradicting to $\alpha^{(i)} \leq \alpha < \frac{3}{2}$. As a result, we can further assume $|T_1| = 1$. By the first point of Lemma 2.8, we have $\text{MMS}_i(n, E) \geq c_i(T_1)$ and accordingly, $\frac{c_i(A_i)}{\text{MMS}_i(n,E)} \leq \frac{c_i(A_i)}{c_i(T_1)} = \alpha^{(i)} \leq \alpha$. For $1 < \alpha < \frac{3}{2}$ and $n \geq 3$, it is not hard to verify that $\alpha \leq \frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$, completing the proof for this case.

Case 2: $\forall j \neq i$, $\text{MMS}_i(2, A_i \cup A_j) > \text{MMS}_i(2, A_i)$ holds. According to Lemma 4.9, for any $j \neq i$, the following holds

$$\text{MMS}_i(2, A_i \cup A_j) \leq \frac{1}{2}c_i(A_i) + c_i(A_j). \tag{5}$$

Due to the construction of $\alpha^{(i)}$, for any $j \neq i$, we have $c_i(A_i) \leq \alpha^{(i)} \cdot \text{MMS}_i(2, A_i \cup A_j)$. Combining Inequality (5), we have $c_i(A_j) \geq \frac{2-\alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)$ for any $j \neq i$. Thus, the following holds,

$$\frac{c_i(A_i)}{\text{MMS}_i(n, E)} \leq \frac{nc_i(A_i)}{c_i(E)} \leq \frac{nc_i(A_i)}{c_i(A_i) + (n-1)\frac{2-\alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)}. \tag{6}$$

The last expression in (6) is monotonically increasing in $\alpha^{(i)}$, and accordingly, we have

$$\frac{c_i(A_i)}{\text{MMS}_i(n, E)} \leq \frac{n\alpha}{\alpha + (n-1)(1 - \frac{\alpha}{2})}.$$

As for the lower bound, consider an instance of n agents with $\frac{n}{2} \in \mathbb{N}^+$ and a set $E = \{e_1, ..., e_{n^2}\}$ of n^2 chores. Agents have identical cost functions. The cost function of agent 1 is as follows: $c_1(e_j) = \alpha$ for $j = 1, ..., n$ and $c_1(e_j) = 2 - \alpha$ for $j = n + 1, ..., n^2$. Now, consider an allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_i = \{e_{(n-1)i+1}, ..., e_{ni}\}$ for $i = 1, ..., n$. Since $\alpha > 1$, it is easy to see that, except for agent 1, no one else will violate the condition of PMMS, and moreover, the approximation guarantee on MMS is determined by agent 1. For agent 1, since $\frac{n}{2} \in \mathbb{N}^+$, $\text{MMS}_1(2, B_1 \cup B_j) = n$ holds for any $j \geq 2$, and due to $c_1(B_1) = n\alpha$, we can claim that the allocation $\mathbf{B}$ is α-PMMS. It is not hard to verify that $\text{MMS}_1(n, E) = \alpha + (n-1)(2-\alpha)$, yielding the ratio $\frac{n\alpha}{\alpha+(n-1)(2-\alpha)}$, completing the proof. □

The motivating example right before Proposition 4.7, unfortunately, only works for the case of $n = 3$. When n becomes larger, an α-PMMS allocation with $\alpha \geq \frac{3}{2}$ is still possible to provide a non-trivial approximation guarantee on the notion of MMS.

We remain to consider the approximation guarantee of MMS for other fairness criteria. Notice that all of EFX, EF1 and PMMS can have non-trivial guarantee for MMS (i.e., better than n-MMS). However, the converse is not true and even the exact MMS does not provide any substantial guarantee for the other three criteria.

Proposition 4.10. *For any* $n \geq 3$, *there exists an* MMS *allocation that is only* 2-PMMS.

Proposition 4.11. *An* MMS *allocation is not necessarily* β-EF1 *or* β-EFX *for any* $\beta \geq 1$.

5 PRICE OF FAIRNESS

After having compared the fairness criteria between themselves, in this section we study the efficiency of these fairness criteria in terms of the price of fairness with respect to social optimality of an allocation.

5.1 Two Agents

We start with the case of two players. Our first result concerns EF1.

Proposition 5.1. *The price of* EF1 *is* 5/4 *when there are two agents.*

According to Propositions 3.4 and 3.6, EF1 implies 2-MMS and $\frac{3}{2}$-PMMS. The following two propositions confirm an intuition — if one relaxes the fairness condition, then less efficiency will be sacrificed.

Proposition 5.2. *The price of 2-*MMS *is 1 when there are two agents.*

The above proposition is implied directly by Lemma 2.9.

Proposition 5.3. *The price of* $\frac{3}{2}$-PMMS *is* $7/6$ *when there are two agents.*

Proof. We first prove the upper bound. Given an instance I, let $\mathbf{O} = (O_1, O_2)$ be an optimal allocation of I. If the allocation $\mathbf{O}$ is already $\frac{3}{2}$-PMMS, we are done. For the sake of contradiction, we assume that agent 1 violates the condition of $\frac{3}{2}$-PMMS in allocation $\mathbf{O}$, i.e., $c_1(O_1) > \frac{3}{2}\mathrm{MMS}_1(2, E)$. Suppose $O_1 = \{e_1, \dots, e_h\}$ and the index satisfies the following rule; $\frac{c_1(e_1)}{c_2(e_1)} \geq \frac{c_1(e_2)}{c_2(e_2)} \geq \cdots \geq \frac{c_1(e_h)}{c_2(e_h)}$. In this proof, for simplicity, we write $L(k) := \{e_1, ..., e_k\}$ for any $1 \leq k \leq h$ and $L(0) = \emptyset$. Then, let s be the index such that $c_1(O_1 \setminus L(s)) \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$ and $c_1(O_1 \setminus L(s-1)) > \frac{3}{2}\mathrm{MMS}_1(2, E)$. In the following, we divide our proof into two cases.

Case 1: $c_1(L(s)) \leq \frac{1}{2}c_1(O_1)$. Consider allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus L(s)$ and $A_2 = O_2 \cup L(s)$. We first show allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. For agent 1, due to the construction of index s, he does not violate the condition of $\frac{3}{2}$-PMMS. As for agent 2, we claim that $c_2(A_2) = 1 - c_2(O_1 \setminus L(s-1)) + c_2(e_s) < \frac{1}{4} + c_2(e_s)$ because $c_2(O_1 \setminus L(s-1)) \geq c_1(O_1 \setminus L(s-1)) > \frac{3}{2}\mathrm{MMS}_1(2, E) \geq \frac{3}{4}$ where the first inequality transition is due to the fact that O_1 is the bundle assigned to agent 1 in the optimal allocation. If $c_2(e_s) < \frac{1}{2}$, then clearly, $c_2(A_2) < \frac{3}{4} \leq \frac{3}{2}\mathrm{MMS}_2(2, E)$. If $c_2(e_s) \geq \frac{1}{2}$, then $c_2(e_s) = \mathrm{MMS}_1(2, E)$ and accordingly, it is not hard to verify that $c_2(A_2) \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$. Thus, $\mathbf{A}$ is a $\frac{3}{2}$-PMMS allocation.

Next, based on allocation $\mathbf{A}$, we derive an upper bound on the price of $\frac{3}{2}$-PMMS. First, by the order of index, $\frac{c_1(L(s))}{c_2(L(s))} \geq \frac{c_1(O_1)}{c_2(O_1)}$ holds, implying $c_2(L(s)) \leq \frac{c_2(O_1)}{c_1(O_1)}c_1(L(s))$. Since $A_1 = O \setminus L(s)$ and $A_2 = O_2 \cup L(s)$, we have the following:

$$
\begin{aligned}
\text{Price of } \frac{3}{2}\text{-PMMS} &\leq 1 + \frac{c_2(L(s)) - c_1(L(s))}{c_1(O_1) + c_2(O_2)} \\
&\leq 1 + \frac{c_1(L(s))(\frac{c_2(O_1)}{c_1(O_1)} - 1)}{c_1(O_1) + c_2(O_2)} \\
&= 1 + \frac{\frac{c_1(L(s))}{c_1(O_1)}(1 - c_2(O_2) - c_1(O_1))}{c_1(O_1) + c_2(O_2)} \\
&\leq 1 + \frac{\frac{1}{2} - \frac{1}{2}(c_1(O_1) + c_2(O_2))}{c_1(O_1) + c_2(O_2)} \\
&\leq 1 - \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6},
\end{aligned}
$$

where the second inequality due to $c_2(L(s)) \leq \frac{c_2(O_1)}{c_1(O_1)}c_1(L(s))$; the third inequality due to the condition of *Case 1*; and the last inequality is because $c_1(O_1) > \frac{3}{2}\mathrm{MMS}_1(2, E) \geq \frac{3}{4}$.

Case 2: $c_1(L(s)) > \frac{1}{2}c_1(O_1)$. We first derive a lower bound on $c_1(e_s)$. Since $c_1(e_s) = c_1(O_1 \setminus L(s-1)) + c_1(L_s) - c_1(O_1)$, combine which with the condition of Case 2 implying $c_1(e_s) > c_1(O_1 \setminus L(s-1)) - \frac{1}{2}c_1(O_1)$, and consequently we have $c_1(e_s) > \frac{3}{2}\mathrm{MMS}_1(2, E) - \frac{1}{2}c_1(O_1) \geq \frac{1}{4}$ where the last transition is due to $\mathrm{MMS}_1(2, E) \geq \frac{1}{2}$ and $c_1(O_1) \leq 1$. Then, we consider two subcases.

If $0 \leq c_2(e_s) - c_1(e_s) \leq \frac{1}{8}$, consider an allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus \{e_s\}$ and $A_2 = O_2 \cup \{e_s\}$. We first show the allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. For agent 1, since $c_1(e_s) > \frac{1}{4}$, $c_1(A_1) = c_1(O_1) - c_1(e_s) < \frac{3}{4} \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$. As for agent 2, $c_2(A_2) = c_2(O_2) + c_2(e_s) \leq 1 - c_1(O_1) + c_2(e_s) < \frac{1}{4} + c_2(e_s)$. If $c_2(e_s) < \frac{1}{2}$, then clearly, $c_2(A_2) \leq \frac{3}{4} < \frac{3}{2}\mathrm{MMS}_2(2, E)$ holds. If $c_2(e_s) \geq \frac{1}{2}$, we have $c_2(e_s) = \mathrm{MMS}_2(2, E)$ and accordingly, it is not hard to verify that $c_2(A_2) \leq \frac{3}{2}\mathrm{MMS}_2(2, E)$. Thus, the allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. Next, based on the allocation $\mathbf{A}$, we derive an upper bound regarding the price of $\frac{3}{2}$-PMMS,

$$
\begin{aligned}
\text{Price of } \frac{3}{2}\text{-PMMS} &\leq \frac{c_1(O_1) - c_1(e_s) + c_2(O_2) + c_2(e_s)}{c_1(O_1) + c_2(O_2)} \\
&\leq 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6},
\end{aligned}
$$

where the second inequality due to $0 \leq c_2(e_s) - c_1(e_s) \leq \frac{1}{8}$ and $c_1(O_1) > \frac{3}{4}$.

If $c_2(e_s) - c_1(e_s) > \frac{1}{8}$, consider an allocation $\mathbf{A}' = (A'_1, A'_2)$ with $A'_1 = \{e_s\}$ and $A'_2 = E \setminus \{e_s\}$. We first show that the allocation $\mathbf{A}'$ is $\frac{3}{2}$-PMMS. For agent 1, due to Lemma 2.8, $c_1(e_s) \leq \mathrm{MMS}_1(2, E)$ holds. As for agent 2, since $c_2(e_s) \geq c_1(e_s) > \frac{1}{4}$, we have $c_2(A'_2) = c_2(E) - c_2(e_s) < \frac{3}{4} \leq \frac{3}{2}\mathrm{MMS}_2(2, E)$. Thus, the allocation $\mathbf{A}'$ is $\frac{3}{2}$-PMMS. In the following, we first derive an upper bound for $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})$, then based on the bound, we provide the target upper bound for the price of fairness. Since $c_1(O_1) > \frac{3}{4}$ and $c_2(e_s) - c_1(e_s) > \frac{1}{8}$, we have $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\}) = c_2(O_1) - c_1(O_1) - (c_2(e_s) - c_1(e_s)) < \frac{1}{8}$, and then, the following holds,

$$
\begin{aligned}
\text{Price of } \frac{3}{2}\text{-PMMS} &\leq 1 + \frac{c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})}{c_1(O_1) + c_2(O_2)} \\
&\leq 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6}.
\end{aligned}
$$

Up to here, we complete the proof of upper bound.

Regarding lower bound, consider an instance I with two agents and a set $E = \{e_1, e_2, e_3, e_4\}$ of four chores. The cost function for agent 1 is: $c_1(e_1) = \frac{3}{8}, c_1(e_2) = \frac{3}{8} + \epsilon, c_1(e_3) = \frac{1}{8} - \epsilon, c_1(e_4) = \frac{1}{8}$ where $\epsilon > 0$ takes arbitrarily small value. For agent 2, here cost function is: $c_2(e_1) = c_2(e_2) = \frac{1}{2}, c_2(e_3) = c_2(e_4) = 0$. It is not hard to verify that $\mathrm{MMS}_i(2, E) = \frac{1}{2}$ for any $i = 1, 2$. In the optimal allocation, the assignment is; e_1, e_2 to agent 1 and e_3, e_4 to agent 2, resulting in $\mathrm{OPT}(I) = \frac{3}{4} + \epsilon$. Observe that to satisfy $\frac{3}{2}$-PMMS, agent 1 cannot receive both chores e_1, e_2, and accordingly, the minimum social cost of a $\frac{3}{2}$-PMMS allocation is $\frac{7}{8}$ by assigning e_1 to agent 1 and the rest chores to agent 2. Based on this instance, when $n = 2$, the price of $\frac{3}{2}$-PMMS is at least $\frac{\frac{7}{8}}{\frac{6}{8}+\epsilon} \to \frac{7}{6}$ as $\epsilon \to 0$. □

We remark that if we have an *oracle* for the maximin share, then our constructive proof of Proposition 5.3 can be transformed into an efficient algorithm for finding a 3/2-PMMS allocation whose cost is at most $\frac{7}{6}$ times the optimal social cost. Moving to other fairness criteria, we have the following uniform bound.

Proposition 5.4. *The price of* PMMS, MMS*, and* EFX *are all 2 when there are two agents.*

5.2 More than Two Agents

Note that the existence of an MMS allocation is not guaranteed in general [7, 31] and the existence of PMMS or EFX allocation is still open when $n \geq 3$. Nonetheless, we are still interested in the prices of fairness in case such a fair allocation does exist. Observe that when the number of chore $m \leq 2$, the price of EF1, EFX, PMMS is trivially 1. If $m = 1$, assigning the unique chore to any agent satisfies all these three fairness criteria, so does the optimal allocation. If $m = 2$, in an optimal allocation, it never happens that both of the two chores are assigned to the same agent. The reason is that if an agent has the smallest cost on one chore, then his cost on another chore is higher than someone else due to the normalized cost function. In the following, we settle down the case of $m \geq 3$.

Proposition 5.5. *For $n \geq 3$ and $m \geq 3$, the price of* EF1, EFX *and* PMMS *are all infinite.*

In the context of goods allocation, Barman et al. [8] present an asymptotically tight price of EF1, $O(\sqrt{n})$. However, as shown by Proposition 5.5, when allocating chores, the price of EF1 is infinite, which shows a sharp contrast between goods and chores allocation.

By using a similar construction to the one in the proof of Proposition 5.5, we can establish the following proposition.

Proposition 5.6. *For $n \geq 3$, the price of $\frac{3}{2}$-*PMMS *is infinite.*

We are now left with MMS fairness. Let us first provide upper and lower bounds on the price of MMS.

Proposition 5.7. *For $n \geq 3$, the price of* MMS *is at most n^2 and at least $\frac{n}{2}$.*

As mentioned earlier, the existence of MMS allocation is not guaranteed. So we also provide an asymptotically tight price of 2-MMS.

Proposition 5.8. *For $n \geq 3$, the price of 2-*MMS *is $\Theta(n)$*

6 CONCLUSIONS

In this paper, we are concerned with fair allocations of indivisible chores among agents under the setting that the cost functions are additive. First we have established pairwise connections between several relaxations of the envy-free fairness in allocating, which look at how an allocation under one fairness criterion provides an approximation guarantee for fairness under another criterion. Some of our results have shown a sharp contrast to what is known in allocating indivisible goods, reflecting the difference between goods and chores allocation. Then we have studied the trade-off between fairness and efficiency, for which we have established the price of fairness for all these fairness notions. We hope our results have provided an almost complete picture for the connections between these chores fairness criteria together with their individual efficiencies relative to social optimum.

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