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Infinite dimensional tensor variational inequalities with applications to an economic equilibrium problem

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ABSTRACT
In this paper, we present a general oligopolistic market equilibrium model in which each firm produces several commodities in a time interval. To this aim, we introduce tensor variational inequalities in Hilbert spaces which are a powerful tool to analyse the model. Indeed we characterize the equilibrium condition by means of a suitable time-dependent tensor variational inequality. In addition, we prove some existence and regularity results and a numerical scheme to compute the solution. Finally we provide a numerical example.

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1. Introduction

In recent years, finite dimensional variational inequalities modelled in the class of tensors have been introduced and studied. Results on existence, uniqueness and regularity of solutions are available (see, for instance, [3,4,15] and the reference therein). This class of inequalities has an important role to study some economic equilibrium problems.

The aim of this paper is to study a general dynamic oligopolistic market equilibrium problem, which is the problem of finding a trade equilibrium in a supply–demand market between a finite number of spatially separated firms which produce several different goods in a time interval and act in a noncooperative behaviour. For this purpose, tensor variational inequalities in Hilbert spaces are introduced and analysed. In particular some existence results, a Minty–Browder-type characterization and some continuity theorems are obtained. The regularity results allow us to introduce a numerical scheme for computing the dynamic variational solution. Thanks to a discretization of the time interval, we are able to use the projection method presented in [6] to solve the static tensor variational inequalities. After that we construct the dynamic solution by using a suitable interpolation. Making use of theoretical arguments, the general dynamic oligopolistic market equilibrium model is examined. It is the time-dependent version of the economic equilibrium problem presented in [3] and extensively studied in [4] for what concerns the ill-posedness and the stability analysis. The introduction of the time in equilibrium models is motivated by the

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fact it allows one to explore the dynamics of adjustment processes in which a delay on time response is operating [14]. For this reason, the model appears as a more realistic generalization of the one presented in [3]. Moreover, we apply the theoretical results to establish the existence and regularity of a dynamic equilibrium solution which allow us to provide a computational procedure to compute such a distribution.

This economic model has been studied intensively in the last years. In [1], the dynamic oligopolistic market equilibrium problem has been presented starting by the time-dependent Cournot–Nash equilibrium principle. In [7] and [2], the behaviour of the market is described through the Lagrange multipliers, by using the infinite dimensional duality theory. In [8] and [9], the model introduced in [1] has been improved in a more realistic way with the addition of production and demand excesses. In [11,12], the model has been analysed from the policymaker's point of view (with the aim to study how the commodity shipment can be controlled by means of the imposition of taxes or incentives) and the regulatory tax definition is formulated by an inverse variational inequality. Different generalizations have been also studied: when the constraint set depends on the expected equilibrium solution and, hence, the equilibrium conditions are expressed by an evolutionary quasi-variational inequality [10,13] or when the uncertainty is considered and consequently the random time-dependent oligopolistic market equilibrium problem is modelled by a stochastic variational inequality [5].

We organize this paper as follows. In Section 2, we prove some existence and continuity results for tensor variational inequalities in infinite dimensional spaces. Moreover, a numerical method is presented and its convergence analysis is discussed. In Section 3, we introduce a time-dependent version of a demand–supply market model and we establish the equivalence between the general dynamic Cournot–Nash equilibrium principle and a suitable evolutionary tensor variational inequality. We prove also existence and regularity results for the dynamic equilibrium distribution. Then, a numerical example is examined. Finally, Section 4 is devoted to some concluding remarks.

2. Tensor variational inequalities in infinite dimensional spaces

This section deals with the introduction and study of tensor variational inequalities in infinite dimensional spaces to analyse a dynamic economic equilibrium model.

First, we recall some definitions on tensors. Let us fix finite dimensional vector spaces $V_i, i = 1, \dots, N$. An N-order tensor is an element of the N-product space $V_1 \times \cdots \times V_N$. Let us denote tensors by italic capital letters $\mathcal{A}, \mathcal{B}, \dots$. A tensor $\mathcal{A}$ of order N is indicated by its entries: the element $(i_1, \dots, i_N)$ of $\mathcal{A}$ is denoted by $a_{i_1,\dots,i_N}$. We note that vectors are tensors of order 1 (denoted usually by small letters $v, w, \dots$) whereas matrices, not necessarily squared, are tensors of order 2 (denoted usually by capital letters $A, B, \dots$). When $V_i = V$, $i = 1, \dots, N$, an N-order tensor on a vector space V of dimension m has m^N entries. In particular, we denote by $\mathbb{R}^{[m_1 \dots m_N]}$ the class of N-order tensors made by $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N}$. Whereas if $m_i = m, i = 1, \dots, N$, we indicate by $\mathbb{R}^{[N,m]}$ the set of all N-order m-dimensional real tensors. When we fix all indices except two, we obtain slices (see, for instance, [16]): two-dimensional sections of a tensor. For examples, a third-order tensor has horizontal, lateral and frontal slices.

The vector space $\mathbb{R}^{[N,m]}$ becomes an Hilbert space if we endow it with the inner product $\langle \cdot, \cdot \rangle$ defined as follows.

Definition 2.1: Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[N,m]}$. Let us define the application $\langle \cdot, \cdot \rangle : \mathbb{R}^{[N,m]} \times \mathbb{R}^{[N,m]} \to \mathbb{R}$ as

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1=1}^{m} \dots \sum_{i_N=1}^{m} a_{i_1,\dots,i_N} b_{i_1,\dots,i_N}.$$

Let us denote by $\| \cdot \|$ the norm induced by the inner product $\langle \cdot, \cdot \rangle$.

Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us indicate by $L^p(I, \mathbb{R}^{[N,m]})$ the space of tensor functions $\mathcal{A} : I \to \mathbb{R}^{[N,m]}$ such that

$$\|\mathcal{A}\|^p = \int_I \|\mathcal{A}(s)\|^p \, \mathrm{d}s < +\infty,$$

where $\|\mathcal{A}(s)\|^p = \sum_{i_1=1}^{m} \dots \sum_{i_N=1}^{m} |a_{i_1,\dots,i_N}(s)|^p$. The pairing between the reflexive Banach spaces $L^q(I, \mathbb{R}^{[N,m]})$ and $L^p(I, \mathbb{R}^{[N,m]})$ is denoted by $\ll \cdot, \cdot \gg$ and defined as

$$\ll \mathcal{A}, \mathcal{B} \gg = \int_I \langle \mathcal{A}(s), \mathcal{B}(s) \rangle \, \mathrm{d}s,$$

where $\mathcal{A} \in (L^p(I, \mathbb{R}^{[N,m]}))^* = L^q(I, \mathbb{R}^{[N,m]})$ and $\mathcal{B} \in L^p(I, \mathbb{R}^{[N,m]})$.

Definition 2.2: Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let K be a nonempty, closed and convex subset of $L^p(I, \mathbb{R}^{[N,m]})$ and let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$. An infinite dimensional tensor variational inequality is the problem of finding $\mathcal{X} \in K$ such that

$$\ll F(\mathcal{X}), \mathcal{Y} - \mathcal{X} \gg \geq 0, \quad \forall \mathcal{Y} \in K, \tag{1}$$

namely

$$\int_I \langle F(s, \mathcal{X}(s)), \mathcal{Y}(s) - \mathcal{X}(s) \rangle \, \mathrm{d}s \geq 0, \quad \forall \mathcal{Y} \in K.$$

In particular, if we consider the case $p = q = 2$ and, hence, the Hilbert space $L^2([0, T], \mathbb{R}^{[N,m]})$, an evolutionary tensor variational inequality is the problem of finding $\mathcal{X} \in K$ such that

$$\int_0^T \langle F(t, \mathcal{X}(t)), \mathcal{Y}(t) - \mathcal{X}(t) \rangle \, dt \geq 0, \quad \forall \mathcal{Y} \in K,$$

where K is a nonempty, closed and convex subset of $L^2([0, T], \mathbb{R}^{[N,m]})$ and $F : [0, T] \times K \to L^2([0, T], \mathbb{R}^{[N,m]})$.

For applications, it is very useful the point-to-point equivalent formulation of infinite dimensional tensor variational inequality (1), as the next lemma establishes.

Lemma 2.3: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let K be a nonempty, closed and convex subset of $L^p(I, \mathbb{R}^{[N,m]})$ and let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$. The infinite dimensional tensor variational inequality* (1) *is equivalent to*

$$\langle F(s, \mathcal{X}(s)), \mathcal{Y}(s) - \mathcal{X}(s) \rangle \geq 0, \quad \forall \mathcal{Y}(s) \in K(s), \ a.e. \ in \ I, \tag{2}$$

where $K(s) = \{\mathcal{Y}(s) \in \mathbb{R}^{[N,m]} : \mathcal{Y} \in K\}$, a.e. in I.

Proof: We only have to show that (1) implies (2), since the opposite implication is trivial. Arguing by contradiction, if (2) is false, there exists a subset $J \subset I$ with positive measure such that

$$\exists \bar{\mathcal{Y}} \in K : \langle F(s, \mathcal{X}(s)), \bar{\mathcal{Y}}(s) - \mathcal{X}(s) \rangle < 0, \quad \text{a.e. in } J.$$

Setting

$$\mathcal{Y}(s) = \begin{cases} \mathcal{X}(s), & \text{a.e. in } I \setminus J, \\ \bar{\mathcal{Y}}(s), & \text{a.e. in } J, \end{cases}$$

we have

$$\ll F(\mathcal{X}), \mathcal{Y} - \mathcal{X} \gg = \int_J \langle F(s, \mathcal{X}(s)), \bar{\mathcal{Y}}(s) - \mathcal{X}(s) \rangle \, \mathrm{d}s < 0,$$

which is a contradiction. ■

2.1. Existence results

Some existence results are proved in this section. To this purpose, let us give some preliminary definitions.

Definition 2.4: Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty subset of $L^p(I, \mathbb{R}^{[N,m]})$. A tensor mapping $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ is said to be

- monotone on K if

$$\ll F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \gg \geq 0, \quad \forall \mathcal{X}, \mathcal{Y} \in K;$$

- strictly monotone on K if

$$\ll F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \gg > 0, \quad \forall \mathcal{X}, \mathcal{Y} \in K, \ \mathcal{X} \neq \mathcal{Y};$$

- strongly monotone on K if there exists $\nu > 0$ such that

$$\ll F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \gg \geq \nu \|\mathcal{X} - \mathcal{Y}\|^2, \quad \forall \mathcal{X}, \mathcal{Y} \in K;$$

- pseudomonotone (in the sense of Karamadian) on K if

$$\ll F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \gg \geq 0 \implies \ll F(\mathcal{X}), \mathcal{X} - \mathcal{Y} \gg \geq 0, \quad \forall \mathcal{X}, \mathcal{Y} \in K;$$

- strictly pseudomonotone on K if

$$\ll F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \gg \geq 0 \implies \ll F(\mathcal{X}), \mathcal{X} - \mathcal{Y} \gg > 0, \quad \forall \mathcal{X}, \mathcal{Y} \in K, \ \mathcal{X} \neq \mathcal{Y}.$$

Definition 2.5: Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a convex subset of $L^p(I, \mathbb{R}^{[N,m]})$. A tensor mapping $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ is said to be

- *hemicontinuous along line segments* if the function

$$\xi \mapsto \ll F(\xi \mathcal{X} + (1 - \xi)\mathcal{Y}), \mathcal{W} \gg, \quad \xi \in [0, 1]$$

  is continuous for all $\mathcal{X}, \mathcal{Y}, \mathcal{W} \in K$;

- *continuous on finite dimensional subspaces* if for any finite dimensional subspace M of $L^p(I, \mathbb{R}^{[N,m]})$, with $K \cap M \neq \emptyset$, the restricted operator $F : K \cap M \to L^q(I, \mathbb{R}^{[N,m]})$ is continuous from the norm topology of $K \cap M$ to the weak* topology of $L^q(I, \mathbb{R}^{[N,m]})$.

A preliminary result is the following Minty–Browder-type Lemma.

Lemma 2.6: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty, convex and closed subset of $L^p(I, \mathbb{R}^{[N,m]})$. Let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ be a pseudomonotone and continuous on finite dimensional subspaces tensor mapping. Then $\mathcal{X} \in K$ is a solution to (1) if and only if*

$$\ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{X} \gg \geq 0, \quad \forall \mathcal{Y} \in K. \tag{3}$$

Proof: First we suppose that $\mathcal{X} \in K$ is a solution to (1). Since F is pseudomonotone, it implies

$$\ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{X} \gg \geq 0, \quad \forall \mathcal{Y} \in K.$$

Conversely, taking $\mathcal{X} \in K$ a solution to (3), we consider

$$\mathcal{X}_\theta = \theta \mathcal{Y} + (1 - \theta)\mathcal{X} \in K,$$

for arbitrary $\theta \in]0, 1]$ and $\mathcal{Y} \in K$. Then $\mathcal{X}_\theta \in K$ and, making use of (3), we have $\theta \ll F(\mathcal{X}_\theta), \mathcal{Y} - \mathcal{X} \gg \geq 0$. Hence, we obtain

$$\ll F(\mathcal{X}_\theta), \mathcal{Y} - \mathcal{X} \gg \geq 0. \tag{4}$$

Letting $\theta \to 0^+$, by the continuity of F on finite-dimensional subspaces, we have that $F(\mathcal{X}_\theta)$ weak*-converges to $F(\mathcal{X})$. Taking into account (4), we deduce that $\ll F(\mathcal{X}), \mathcal{Y} - \mathcal{X} \gg \geq 0$. Therefore $\mathcal{X}$ is a solution to (1). ■

We are able to establish the following result which has very mild hypothesis on the tensor mapping F.

Theorem 2.7: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty, weakly compact and convex subset of $L^p(I, \mathbb{R}^{[N,m]})$. Let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ be a pseudomonotone and continuous on finite dimensional subspaces tensor mapping. Then tensor variational inequality (1) admits at least a solution.*

Proof: Let A be a finite dimensional subspace of $L^p(I, \mathbb{R}^{[N,m]})$ such that $A \cap K$ is nonempty. Let us introduce the injection map

$$P_A : A \hookrightarrow L^p(I, \mathbb{R}^{[N,m]})$$

and its adjoint

$$P_A^* : L^q(I, \mathbb{R}^{[N,m]}) \to A^*.$$

Then the map $P_A^* F P_A$ from $A \cap K$ into A^* is continuous. Since K is weakly compact and, hence, bounded, the set $K \cap A$ is closed, bounded and convex in A. Moreover, since A

is finite dimensional, without loss of generality, we may assume $A = \mathbb{R}^{[N,m]}$ and we may identify A^* with A. Then there exists $\mathcal{X}_A \in A \cap K$ such that

$$\ll P_A^* F P_A(\mathcal{X}_A), \mathcal{X} - \mathcal{X}_A \gg \geq 0, \quad \forall \mathcal{X} \in A \cap K.$$

From the previous inequality, it follows

$$\ll F(\mathcal{X}_A), \mathcal{X} - \mathcal{X}_A \gg \geq 0, \quad \forall \mathcal{X} \in A \cap K.$$

By virtue of Lemma 2.6, we have

$$\ll F(\mathcal{X}), \mathcal{X} - \mathcal{X}_A \gg \geq 0, \quad \forall \mathcal{X} \in A \cap K. \tag{5}$$

Now, for all $\mathcal{Y} \in K$, we define

$$S(\mathcal{Y}) = \{\mathcal{X} \in K : \ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{X} \gg \geq 0\}.$$

The family $\{S(\mathcal{Y}) : \ \mathcal{Y} \in K\}$ has the finite intersection property. Indeed, for any finite family of subsets $\{\mathcal{X}_i\}_{1 \leq i \leq m}$ of K, let A be the finite dimensional subspace spanned by $\{\mathcal{X}_i\}_{1 \leq i \leq m}$. By the finite dimensional case, (5) has a solution $\mathcal{X}_A$. Then, in particular, we obtain

$$\ll F(\mathcal{X}_i), \mathcal{X}_i - \mathcal{X}_A \gg \geq 0, \quad \forall\, 1 \leq i \leq m.$$

Consequently, $\mathcal{X}_A \in \bigcap_{i=1}^m S(\mathcal{X}_i)$. Then $S(\mathcal{Y})$ is nonempty, for every $\mathcal{Y} \in K$. Since $S(\mathcal{Y})$ is weakly closed, for all $\mathcal{Y} \in K$, and K is weakly compact, it follows that $\bigcap_{\mathcal{X} \in K} S(\mathcal{X})$ is nonempty. Choosing $\mathcal{Z} \in \bigcap_{\mathcal{X} \in K} S(\mathcal{X})$ it results

$$\ll F(\mathcal{Z}), \mathcal{X} - \mathcal{Z} \gg \geq 0, \quad \forall \mathcal{X} \in K,$$

which concludes the proof. ■

It is possible to weaken the hypothesis in the previous theorem assuming that the tensor mapping F is not continuous on finite dimensional subspace but only hemicontinuous along line segments. Precisely we have:

Theorem 2.8: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty, weakly compact and convex subset of $L^p(I, \mathbb{R}^{[N,m]})$. Let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ be a pseudomonotone and hemicontinuous along line segments tensor mapping. Then tensor variational inequality* (1) *admits at least a solution.*

Proof: Let us consider the following nonempty sets:

$$S(\mathcal{Y}) = \{\mathcal{X} \in K : \ll F(\mathcal{X}), \mathcal{Y} - \mathcal{X} \gg \geq 0\}$$

$$M(\mathcal{Y}) = \{\mathcal{X} \in K : \ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{X} \gg \geq 0\}.$$

As first step, we show that $\bigcap_{\mathcal{Y} \in K} M(\mathcal{Y}) \neq \emptyset$. Since F is pseudomonotone, then $S(\mathcal{Y}) \subset M(\mathcal{Y})$. Moreover, since $M(\mathcal{Y})$ is closed, it follows that

$$\overline{S(\mathcal{Y})} \subset M(\mathcal{Y}). \tag{6}$$

Let $\{\mathcal{Y}_1, \dots, \mathcal{Y}_n\}$ be a finite subset of K and let $\mathcal{Y} \in \text{conv}(\{\mathcal{Y}_1, \dots, \mathcal{Y}_n\})$, which means

$$\mathcal{Y} = \sum_{i=1}^{n} \alpha_i \mathcal{Y}_i, \quad \text{with} \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1.$$

Assume by contradiction that $\mathcal{Y} \notin \bigcup_{i=1}^{n} \overline{S(\mathcal{Y}_i)}$. Then, we get

$$\ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{Y}_i \gg < 0, \quad \forall i = 1, \dots, n.$$

We have

$$\sum_{i=1}^{n} \alpha_i \ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{Y}_i \gg = \ll F(\mathcal{Y}), \mathcal{Y} - \sum_{i=1}^{n} \alpha_i \mathcal{Y}_i \gg < 0,$$

which is a contradiction. Thus $\text{conv}(\{\mathcal{Y}_1, \dots, \mathcal{Y}_n\}) \subset \bigcup_{i=1}^{n} \overline{S(\mathcal{Y}_i)}$ and, since for arbitrary $\mathcal{Y}_0 \in K$ the set $\overline{S(\mathcal{Y}_0)}$ is compact, it results that $\bigcap_{\mathcal{Y} \in K} \overline{S(\mathcal{Y})} \neq \emptyset$. Therefore, by (6), we deduce that $\bigcap_{\mathcal{Y} \in K} M(\mathcal{Y}) \neq \emptyset$. An element $\mathcal{Z} \in \bigcap_{\mathcal{Y} \in K} M(\mathcal{Y}) \neq \emptyset$ is such that

$$\ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{Z} \gg \geq 0.$$

By using same arguments of Lemma 2.6, we conclude that $\mathcal{Z}$ is the desired solution. ■

Remark 2.1: Let us note that if F is in addition strictly pseudomonotone, then the solution to the infinite tensor variational inequality is unique. Indeed, let us suppose that (1) has two solutions $\mathcal{X}_1, \mathcal{X}_2 \in K$ such that $\mathcal{X}_1 \neq \mathcal{X}_2$, that is

$$\ll F(\mathcal{X}_1), \mathcal{Y} - \mathcal{X}_1 \gg \geq 0, \quad \forall \mathcal{Y} \in K, \tag{7}$$

and

$$\ll F(\mathcal{X}_2), \mathcal{Y} - \mathcal{X}_2 \rangle \gg \geq 0, \quad \forall \mathcal{Y} \in K. \tag{8}$$

We write (7) with $\mathcal{Y} = \mathcal{X}_2$ and (8) with $\mathcal{Y} = \mathcal{X}_1$:

$$\ll F(\mathcal{X}_1), \mathcal{X}_2 - \mathcal{X}_1 \gg \geq 0, \tag{9}$$

$$\ll F(\mathcal{X}_2), \mathcal{X}_1 - \mathcal{X}_2 \gg \geq 0. \tag{10}$$

Taking into account the strict pseudomonotonicity of F and (9), it results

$$\ll F(\mathcal{X}_2), \mathcal{X}_2 - \mathcal{X}_1 \gg > 0,$$

which is in contradiction with (10).

Now we deal with the case in which the set K is not weakly compact. In this case is necessary an additional coercivity condition, as in the following result.

Theorem 2.9: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty, convex and closed subset of $L^p(I, \mathbb{R}^{[N,m]})$. Let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ be a pseudomonotone and hemicontinuous along line segments tensor mapping. Suppose also that there exists a nonempty, weakly compact and convex subset C of K such that for every $\mathcal{X} \in K \setminus C$ there exists $\mathcal{Z} \in C$ with $\ll F(\mathcal{X}), \mathcal{X} - \mathcal{Z} \gg > 0$. Then, tensor variational inequality (1) admits at least a solution.*

Proof: Let $A = \{\mathcal{Y}_1, \dots, \mathcal{Y}_n\}$ be a finite subset of K and let us consider the convex subset $C_1 = \mathrm{conv}(A \cup C)$, which is weakly compact since C is a convex and weakly compact subset of K. Then taking into account Theorem 2.8, there exists $\overline{\mathcal{X}} \in C_1$ such that

$$\ll F(\overline{\mathcal{X}}), \mathcal{Y} - \overline{\mathcal{X}} \gg \geq 0, \quad \forall \mathcal{Y} \in C_1. \tag{11}$$

We note that $\overline{\mathcal{X}} \in C$. Otherwise if $\overline{\mathcal{X}} \in K \setminus C$ and making use of the coercivity assumption, we obtain a contradiction with (11). Consider now the following nonempty sets:

$$S(\mathcal{Y}) = \{\mathcal{X} \in K : \ll F(\mathcal{X}), \mathcal{Y} - \mathcal{X} \gg \geq 0\},$$

$$M(\mathcal{Y}) = \{\mathcal{X} \in K : \ll F(\mathcal{Y}), \mathcal{Y} - \mathcal{X} \gg \geq 0\}.$$

By using (11), it follows that $\bigcap_{i=1}^n S(\mathcal{Y}_i) \neq \emptyset$ and, therefore, $\bigcap_{i=1}^n \overline{S(\mathcal{Y}_i)} \neq \emptyset$. Since F is pseudomonotone, it follows

$$\emptyset \neq \bigcap_{i=1}^n \overline{S(\mathcal{Y}_i)} \subset \bigcap_{i=1}^n M(\mathcal{Y}_i).$$

Thus the family of closed subsets $\{M(\mathcal{Y})\}_{\mathcal{Y} \in K}$ has the finite intersection property. By the weak compactness of C, we have that $\bigcap_{i=1}^n M(\mathcal{Y}_i) \neq \emptyset$. By virtue of Lemma 2.6, $\overline{\mathcal{X}} \in \bigcap_{i=1}^n M(\mathcal{Y}_i)$ is also a solution to (1). ∎

Finally we remark that if F is a Carathéodory function such that

$$\|F(s, \mathcal{X}(s))\|^q \leq \alpha(s) + \|\mathcal{X}(s)\|^p, \quad \forall \mathcal{X} \in K, \text{ a.e. in } I,$$

where $\alpha \in L^1(I, \mathbb{R})$, then it is hemicontinuous along line segments. Indeed for each sequence $\{\lambda_r\}$ such that $\lambda_r \to \lambda \in [0, 1]$, as $r \to +\infty$, and for every $\mathcal{X}, \mathcal{Y} \in K$, it results

$$\lim_r \int_I \|F(s, \lambda_r \mathcal{X}(s) + (1 - \lambda_r)\mathcal{Y}(s)) - F(s, \lambda \mathcal{X}(s) + (1 - \lambda)\mathcal{Y}(s))\|^q \mathrm{d}s = 0,$$

and, hence,

$$\lim_r \int_I \langle F(s, \lambda_r \mathcal{X}(s) + (1 - \lambda_r)\mathcal{Y}(s)), \mathcal{X}(s) - \mathcal{Y}(s) \rangle \,\mathrm{d}s$$
$$= \int_I \langle F(s, \lambda \mathcal{X}(s) + (1 - \lambda)\mathcal{Y}(s)), \mathcal{X}(s) - \mathcal{Y}(s) \rangle \,\mathrm{d}s.$$

2.2. Continuity results

In this section, we prove continuity results for tensor variational inequalities. A theoretical concept we use in the following is the set convergence introduced by Kuratowski in the 1960s [17] for a set sequence in a given metric space (X, d).

Let $\{K_r\}$ be a sequence of subsets of X. Recall that

$$d - \underline{\lim}_r K_r = \{x \in X : \exists \{x_r\} \text{ eventually in } K_r \text{ such that } x_r \to x\}$$

and

$$d - \overline{\lim}_r K_r = \{x \in X : \exists \{x_r\} \text{ frequently in } K_r \text{ such that } x_r \to x\},$$

where *eventually* means that there exists $\delta \in \mathbb{N}$ such that $x_r \in K_r$, for any $r \geq \delta$, and *frequently* means that there exists an infinite subset $N \subseteq \mathbb{N}$ such that $x_r \in K_r$, for every $r \in N$ (in this last case, according to the notation given above, we also write that there exists a subsequence $\{x_{k_r}\} \subseteq \{x_r\}$ such that $x_{k_r} \in K_{k_r}$, for every $r \in \mathbb{N}$).

By definitions, it is easy to verify that $d - \underline{\lim}_r K_r \subseteq d - \overline{\lim}_r K_r$. Now we can present the set convergence in Kuratowski's sense.

Definition 2.10: We say that $\{K_r\}$ converges to some subset $K \subseteq X$ in Kuratowski's sense if and only if $d - \underline{\lim}_r K_r = d - \overline{\lim}_r K_r = K$.

We observe that the set convergence in Kuratowski's sense can also be shown verifying the following conditions:

(K1) for any $x \in K$, there exists a sequence $\{x_r\}$ strongly converging to $x \in X$ such that $x_r \in K_r$, for every $r \in \mathbb{N}$,
(K2) for any subsequence $\{x_r\}$ converging to $x \in X$ such that $x_r \in K_r$, for every $r \in \mathbb{N}$, then the limit x belongs to K.

Let us say that a nonempty subset K of $L^p(I, \mathbb{R}^{[N,m]})$ verifies the Kuratowski convergence property if and only if for every $t \in I$ and every $\{t_r\} \subseteq I$ such that $t_r \to t$, as $r \to +\infty$, the sequence $\{K(t_r)\}$, where $K(t_r) = \{\mathcal{Y}(t_r) \in \mathbb{R}^{[N,m]} : \mathcal{Y} \in K\}$, converges to $K(t)$ in Kuratowski's sense.

Now, we are able to establish the continuity results for evolutionary tensor variational inequality (2) under the strong monotonicity assumption on the tensor mapping F.

Theorem 2.11: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty, convex and closed subset of $L^p(I, \mathbb{R}^{[N,m]})$ verifying the Kuratowski convergence property. Let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ be a continuous and strongly monotone tensor mapping. Then the solution to tensor variational inequality (2) is continuous in I.*

Proof: The existence of a unique solution $\mathcal{X}(t)$ to (2) is ensured by Theorem 2.8 and Remark 2.1. Let us fix $t \in I$ and a sequence $\{t_r\} \subseteq I$ such that $t_r \to t$, as $r \to +\infty$. Let

$\mathcal{X}(t_r)$ be the unique solution to the tensor variational inequality

$$\langle F(t_r, \mathcal{X}(t_r)), \mathcal{Y}(t_r) - \mathcal{X}(t_r) \rangle \geq 0, \quad \forall \mathcal{Y}(t_r) \in K(t_r). \tag{12}$$

We have to verify that $\mathcal{X}(t_r) \to \mathcal{X}(t)$, as $r \to +\infty$. Taking into account the type Minty–Browder Lemma, for any $s \in I$ we have

$$\langle F(s, \mathcal{Y}(s)), \mathcal{Y}(s) - \mathcal{X}(s) \rangle \geq 0, \quad \forall \mathcal{Y}(s) \in K(s).$$

By using $(K1)$ applied to $\mathcal{X}(t) \in K(t)$, there exists a sequence $\{\mathcal{Z}(t_r)\}$ such that $\mathcal{Z}(t_r) \in K(t_r)$, for r large enough, and $\mathcal{Z}(t_r) \to \mathcal{X}(t)$, as $r \to +\infty$. Moreover, we derive that $F(t_r, \mathcal{Z}(t_r)) \to F(t, \mathcal{X}(t))$, as $r \to +\infty$, by the continuity of F. Setting, for r large enough, $\mathcal{Y}(t_r) = \mathcal{Z}(t_r)$ in (12), we have

$$\langle F(t_r, \mathcal{X}(t_r)), \mathcal{Z}(t_r) - \mathcal{X}(t_r) \rangle \geq 0.$$

Making use of the strongly monotonicity assumption, we have

$$\nu \|\mathcal{X}(t_r) - \mathcal{Z}(t_r)\|^2 \leq -\langle F(t_r, \mathcal{Z}(t_r)), \mathcal{X}(t_r) - \mathcal{Z}(t_r) \rangle \leq \|F(\mathcal{Z}(t_r))\| \|\mathcal{X}(t_r) - \mathcal{Z}(t_r)\|$$

and, consequently,

$$\nu \|\mathcal{X}(t_r) - \mathcal{Z}(t_r)\| \leq \|F(\mathcal{Z}(t_r))\|.$$

Hence, it results

$$\|\mathcal{X}(t_r)\| \leq \|\mathcal{X}(t_r) - \mathcal{Z}(t_r)\| + \|\mathcal{Z}(t_r)\| \leq \frac{\|F(t_r, \mathcal{Z}(t_r))\|}{\nu} + \|\mathcal{Z}(t_r)\|.$$

As a consequence, we have that $\{\mathcal{X}(t_r)\}$ is bounded. Therefore there exists $\mathcal{W} \in \mathbb{R}^{[N,m]}$ and there exists a subsequence denoted again by $\{\mathcal{X}(t_r)\}$, such that $\mathcal{X}(t_r) \in K(t_r)$ and $\mathcal{X}(t_r) \to \mathcal{W}$. Making use of (K2), we obtain that $\mathcal{W} \in K(t)$. We show that $\mathcal{W} = \mathcal{X}(t)$. Applying again the type Minty–Browder Lemma, we get

$$\langle F(t_r, \mathcal{Y}(t_r)), \mathcal{Y}(t_r) - \mathcal{X}(t_r) \rangle \geq 0, \quad \forall \mathcal{Y}(t_r) \in K(t_r).$$

By using $(K1)$ once more again for any $\mathcal{Y}(t) \in K(t)$, there exists $\{\mathcal{V}(t_r)\}$ such that $\mathcal{V}(t_r) \in K(t_r)$, for r large enough, and $\mathcal{V}(t_r) \to \mathcal{Y}(t)$. Then, we obtain

$$\langle F(t_r, \mathcal{V}(t_r)), \mathcal{V}(t_r) - \mathcal{X}(t_r) \rangle \geq 0.$$

Passing to the limit as $r \to +\infty$, it follows

$$\langle F(t, \mathcal{Y}(t)), \mathcal{Y}(t) - \mathcal{W} \rangle \geq 0, \quad \forall \mathcal{Y}(t) \in K(t).$$

From the Minty–Browder Lemma again, it results

$$\langle F(t, \mathcal{W}), \mathcal{Y}(t) - \mathcal{W} \rangle \geq 0, \quad \forall \mathcal{Y}(t) \in K(t).$$

From the uniqueness of the solution to (2) we deduce that $\mathcal{W} = \mathcal{X}(t)$ and that $\mathcal{X}(t_r) \to \mathcal{X}(t)$. Finally, since F is continuous, by the inequality

$$\nu \|\mathcal{X}(t_r) - \mathcal{Z}(t_r)\|^2 \leq \langle F(t_r, \mathcal{Z}(t_r)), \mathcal{Z}(t_r) - \mathcal{X}(t_r) \rangle,$$

and the fact that $(\mathcal{X}(t_r) - \mathcal{Z}(t_r)) \to 0$, the claim holds. ∎

We establish now an analogous result for tensor variational inequalities under the milder hypothesis of strict monotonicity on F. We show the following continuity result.

Theorem 2.12: *Let $I \subset \mathbb{R}$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and K be a nonempty, convex, bounded and closed subset of $L^p(I, \mathbb{R}^{[N,m]})$ verifying the Kuratowski convergence property. Let $F : I \times K \to L^q(I, \mathbb{R}^{[N,m]})$ be a continuous and strictly monotone tensor mapping. Then the solution to tensor variational inequality (2) is continuous in I.*

Proof: Let $\{t_r\}$ be a sequence in I such that $t_r \to t$, as $r \to +\infty$. Let $\mathcal{X}(t)$ be the solution to tensor variational inequality (2) and $\mathcal{X}(t_r)$, $\forall r \in \mathbb{N}$, be the solutions to the following tensor variational inequalities:

$$\langle F(t_r, \mathcal{X}(t_r)), \mathcal{Y}(t_r) - \mathcal{X}(t_r)\rangle \geq 0, \quad \forall \mathcal{Y}(t_r) \in K(t_r),\ \forall r \in \mathbb{N}. \tag{13}$$

Fixed $\varepsilon > 0$, let $\mathcal{X}_\varepsilon(t)$ be the unique solution to the following perturbed strongly monotone tensor variational inequality

$$\langle F(t, \mathcal{X}_\varepsilon(t)) + \varepsilon \mathcal{X}_\varepsilon(t), \mathcal{Y}(t) - \mathcal{X}_\varepsilon(t)\rangle \geq 0, \quad \forall \mathcal{Y}(t) \in K(t),\ \text{in } I. \tag{14}$$

Taking into account Theorem 2.11, it follows that $\mathcal{X}_\varepsilon(t)$ is continuous in I. Hence the sequence of solutions $\mathcal{X}_\varepsilon(t_r)$, for every $r \in \mathbb{N}$, to the following tensor variational inequalities:

$$\langle F(t_r, \mathcal{X}_\varepsilon(t_r)) + \varepsilon \mathcal{X}_\varepsilon(t_r), \mathcal{Y}(t_r) - \mathcal{X}_\varepsilon(t_r)\rangle \geq 0, \quad \forall \mathcal{Y}(t_r) \in K(t_r),\ \forall r \in \mathbb{N} \tag{15}$$

converges to $\mathcal{X}_\varepsilon(t)$, as $r \to +\infty$. Furthermore, we note that $\mathcal{X}_\varepsilon(t) \to \mathcal{X}(t)$, as $\varepsilon \to 0$, in I. Indeed, considering $\mathcal{Y}(t) = \mathcal{X}_\varepsilon(t)$, in I, in (2) and $\mathcal{Y}(t) = \mathcal{X}(t)$, in I, in (14) and adding the inequalities, we have

$$\langle F(t, \mathcal{X}(t)) - F(t, \mathcal{X}_\varepsilon(t)), \mathcal{X}_\varepsilon(t) - \mathcal{X}(t)\rangle + \varepsilon\langle \mathcal{X}_\varepsilon(t), \mathcal{X}(t) - \mathcal{X}_\varepsilon(t)\rangle \geq 0, \quad \text{in } I. \tag{16}$$

For the strict monotonicity assumption on F, it results

$$\langle F(t, \mathcal{X}(t)) - F(t, \mathcal{X}_\varepsilon(t)), \mathcal{X}_\varepsilon(t) - \mathcal{X}(t)\rangle < 0, \quad \text{in } I.$$

Therefore, by (16), we obtain

$$\varepsilon\langle \mathcal{X}_\varepsilon(t), \mathcal{X}(t) - \mathcal{X}_\varepsilon(t)\rangle \geq 0, \quad \text{in } I,$$

and, then,

$$\|\mathcal{X}_\varepsilon(t)\|^2 \leq \langle \mathcal{X}_\varepsilon(t), \mathcal{X}(t)\rangle \leq \|\mathcal{X}(t)\|\,\|\mathcal{X}_\varepsilon(t)\|, \quad \text{in } I.$$

Hence, we deduce

$$\|\mathcal{X}_\varepsilon(t)\| \leq \|\mathcal{X}(t)\|, \quad \text{in } I.$$

Since $\mathcal{X}(t) \in K(t)$ and K is a bounded subset of $L^p(I, \mathbb{R}^{[N,m]})$, it results

$$\|\mathcal{X}_\varepsilon(t)\| \leq C, \quad \forall \varepsilon > 0,\ \text{in } I.$$

Then, there exists a subsequence denoted again by $\{\mathcal{X}_\varepsilon(t)\}$ converging in $\mathbb{R}^{[N,m]}$ to an element $\overline{\mathcal{X}}(t) \in K(t)$, in I. We need to show that

$$\overline{\mathcal{X}}(t) = \mathcal{X}(t), \quad \text{in } I.$$

To this aim, taking into account the continuity of F and passing to the limit as $\varepsilon \to 0$ in (14), we obtain

$$\langle F(t, \overline{\mathcal{X}}(t)), \mathcal{Y}(t) - \overline{\mathcal{X}}(t)\rangle \geq 0, \quad \forall \mathcal{Y}(t) \in K(t), \text{ in } I. \tag{17}$$

Then $\overline{\mathcal{X}}$ is a solution to (2), in I. Since the solution to (2) is unique, then the $\mathcal{X}_\varepsilon(t) \to \mathcal{X}(t)$, as $\varepsilon \to 0$, in I.

By repeating the same arguments with $\mathcal{Y}(t_r) = \mathcal{X}(t_r)$, $\forall r \in \mathbb{N}$, in (15), and $\mathcal{Y}(t_r) = \mathcal{X}_\varepsilon(t_r)$, for every $r \in \mathbb{N}$, in (13), we obtain that there exists a subsequence still denoted by $\{\mathcal{X}(t_r)\}$, with $\mathcal{X}(t_r) \in K(t_r)$, for every $r \in \mathbb{N}$, converging to $\overline{\mathcal{X}}(t)$ in $\mathbb{R}^{[N,m]}$, namely $\mathcal{X}(t_r) \to \overline{\mathcal{X}}(t)$, as $r \to +\infty$. Furthermore, by (13), we have

$$\langle F(t, \overline{\mathcal{X}}(t)), \mathcal{Y}(t) - \overline{\mathcal{X}}(t)\rangle \geq 0, \quad \forall \mathcal{Y}(t) \in K(t),$$

and, for the uniqueness of the solution to (2), we deduce

$$\lim_{r \to +\infty} \mathcal{X}(t_r) = \mathcal{X}(t). \qquad \blacksquare$$

We would like to underline that the previous results hold also for evolutionary tensor variational inequalities. In such a case, we obtain the continuity with respect to the time variable.

2.3. Computational procedure

The continuity results allow us to provide a numerical method, by using a discretization procedure, for the calculation of solutions to evolutionary tensor variational inequalities. Under the assumptions of Theorem 2.12, the solution belongs to $C^0([0, T], \mathbb{R}^{[N,m]})$. As a consequence, we can write (2) with $I = [0, T]$ and $p = q = 2$ as

$$\langle F(t, \mathcal{X}(t)), \mathcal{Y}(t) - \mathcal{X}(t)\rangle \geq 0, \quad \forall \mathcal{Y}(t) \in K(t), \text{ in } [0, T]. \tag{18}$$

We describe now a procedure to compute the solution to tensor variational inequality (18) by discretizing the time interval. In detail, we consider a partition of $[0, T]$ such that

$$0 = t_0 < t_1 < \ldots < t_i < \ldots < t_N = T.$$

For each value t_i, $i = 0, 1, \ldots, N$, we make use of the projection method presented in [6] to solve the static tensor variational inequalities

$$\langle F(t_i, \mathcal{X}(t_i)), \mathcal{Y}(t_i) - \mathcal{X}(t_i)\rangle \geq 0, \quad \forall \mathcal{Y}(t_i) \in K(t_i), \ i = 0, 1, \ldots, N.$$

Precisely, starting from any $\mathcal{X}_0(t_i)$ fixed, iteratively $\mathcal{X}(t_i)$ updates according to the formula

$$\mathcal{X}_{k+1}(t_i) = P_{K(t_i)}(\mathcal{X}_k(t_i) - \alpha F(t_i, \mathcal{X}_k(t_i))),$$

for $k \in \mathbb{N}$, where $P_{K(t_i)}(\cdot)$ is the orthogonal projection map onto $K(t_i)$ and α is a suitable chosen positive step length. We remark that $P_{K(t_i)}(\mathcal{X}_k(t_i) - \alpha F(t_i, \mathcal{X}_k(t_i)))$ is the solution of the following quadratic programming problem:

$$\min_{\mathcal{Y}(t_i)\in K(t_i)} \frac{1}{2}\langle \mathcal{Y}(t_i), \mathcal{Y}(t_i)\rangle - \langle \mathcal{X}_k(t_i) - \alpha F(t_i, \mathcal{X}_k(t_i)), \mathcal{Y}(t_i)\rangle,$$

for $k \in \mathbb{N}$. Furthermore $\mathcal{X}^*$ is a solution to the tensor variational inequality if and only if

$$\mathcal{X}^*(t_i) = P_{K(t_i)}(\mathcal{X}^*(t_i) - \alpha F(t_i, \mathcal{X}^*(t_i))).$$

An accurate analysis on the convergence of such a method has been investigated in [6]. This method gives us the solution of each point-to-point variational problems. To obtain the solution in the time interval $[0, T]$, the next step is to interpolate, in a suitable way, such static solutions.

Let us introduce a sequence $\{\pi_r\}$ of partitions (made up of not necessarily equidistant points) of the time interval $[0, T]$ such that $\pi_r = \{t_r^0, t_r^1, \dots, t_r^{N_r}\}$, with $0 = t_r^0 < t_r^1 < \dots < t_r^{N_r} = T$, assuming that

$$k_r = \max\{t_r^s - t_r^{s-1} : s = 1, 2, \dots, N_r\},$$

approaches zero, as $r \to +\infty$.

We construct then the numerical solution to (18) by considering piecewise constant functions, as below

$$\mathcal{X}_r(t) = \sum_{s=1}^{N_r} \mathcal{X}(t_r^s) \mathbb{1}_{[t_r^{s-1}, t_r^s[}(t), \tag{19}$$

where $\mathcal{X}(t_r^s)$ is the solution to (18) for $t = t_r^s$ and $\mathbb{1}_{[t_r^{s-1}, t_r^s[}$ is the characteristic function of the interval $[t_r^{s-1}, t_r^s[$, namely

$$\mathbb{1}_{[t_r^{s-1}, t_r^s[}(t) = \begin{cases} 1 & t \in [t_r^{s-1}, t_r^s[\\ 0 & t \notin [t_r^{s-1}, t_r^s[\end{cases}.$$

We prove that such a sequence converges in L^1 to the solution to (18). Indeed, let us estimate the following integral:

$$\begin{aligned} &\int_0^T \left\| \mathcal{X}(t) - \sum_{s=1}^{N_r} \mathcal{X}(t_r^s) \mathbb{1}_{[t_r^{s-1}, t_r^s[}(t) \right\| dt \\ &= \int_0^T \left\| \sum_{s=1}^{N_r} \mathcal{X}(t) \mathbb{1}_{[t_r^{s-1}, t_r^s[}(t) - \sum_{s=1}^{N_r} \mathcal{X}(t_r^s) \mathbb{1}_{[t_r^{s-1}, t_r^s[}(t) \right\| dt \\ &\le \sum_{s=1}^{N_r} \int_{t_r^{s-1}}^{t_r^s} \|\mathcal{X}(t) - X(t_r^s)\| dt. \end{aligned}$$

Being $\mathcal{X}$ uniformly continuous, we deduce that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $t \in [t_r^{s-1}, t_r^s]$ satisfies the condition $|t - t_r^s| < \delta$ it results

$$\|\mathcal{X}(t) - \mathcal{X}(t_r^s)\| < \frac{\varepsilon}{T}, \quad \forall\, s = 1, 2, \dots, N_r,\ \forall r \in \mathbb{N}.$$

We choose now r large enough such that $k_r < \delta$, hence, we obtain

$$\int_0^T \left\| \mathcal{X}(t) - \sum_{s=1}^{N_r} \mathcal{X}(t_r^s) \mathbb{1}_{[t_r^{s-1}, t_r^s[}(t) \right\| dt < \sum_{s=1}^{N_r} \frac{\varepsilon}{T}(t_r^s - t_r^{s-1}) = \varepsilon. \tag{20}$$

The last estimate implies that sequence (19) converges in L^1-sense to the solution to evolutionary tensor variational inequality (18).

3. The general dynamic oligopolistic market equilibrium model

We are going to present an oligopolistic market equilibrium model in which each firm produces several different commodities and acts in a time interval. Due to the introduction of the time dependence, the new model discussed in this section is an extension of the one introduced in [3]. Moreover, since each firm produces several commodities, it is a generalization of the model analysed in [1].

Let us consider m firms P_i, $i = 1, \dots, m$, and n demand markets Q_j, $j = 1, \dots, n$, which are generally spatially separated. Assume that the commodities, produced by the m firms and consumed by the n markets, are involved during a period of time $[0, T]$, $T > 0$. Let us suppose that every firm P_i produces l different commodities. Let us indicate by $x_{ij}^k(t)$ the commodities of kind k shipment between the firm P_i and the markets Q_j at the time $t \in [0, T]$, $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, l$. Let us indicate by $p_i^k(t)$ the commodity output of kind k produced by the firm P_i, at the time $t \in [0, T]$, $i = 1, \dots, m$, $k = 1, \dots, l$. Let us indicate by $q_j^k(t)$ the demand for the commodity of kind k of the demand market Q_j, at the time $t \in [0, T]$, $j = 1, \dots, n$, $k = 1, \dots, l$. The variables $x_{ij}^k(t)$, $p_i^k(t)$ and $q_j^k(t)$ are nonnegative, for every $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, l$, a.e. in $[0, T]$. For technical reasons, we assume that $\mathcal{X} = (x_{ij}^k) \in L^2([0, T], \mathbb{R}^{[nml]})$.

We suppose also that the commodity shipment of kind k between the producer P_i and the market Q_j has to satisfy time-dependent capacity constraints, namely

$$0 \le \underline{x}_{ij}^k(t) \le x_{ij}^k(t) \le \overline{x}_{ij}^k(t), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \forall k = 1, \dots, l, \ \text{a.e. in } [0, T],$$

where $\underline{\mathcal{X}} = (\underline{x}_{ij}^k)$ and $\overline{\mathcal{X}} = (\overline{x}_{ij}^k)$ are tensor mappings belonging to $L^2([0, T], \mathbb{R}^{[nml]})$. Furthermore, we suppose that the following feasibility conditions hold:

$$p_i^k(t) = \sum_{j=1}^n x_{ij}^k(t), \quad \forall i = 1, \dots, m, \ \forall k = 1, \dots, l, \ \text{a.e. in } [0, T], \tag{21}$$

$$q_j^k(t) = \sum_{i=1}^m x_{ij}^k(t), \quad \forall j = 1, \dots, n, \ \forall k = 1, \dots, l, \ \text{a.e. in } [0, T]. \tag{22}$$

These mean that the quantity produced by each firm P_i of kind k, at time $t \in [0, T]$, must be equal to the sum of the commodities of such kind from that firm to all the demand markets, at the same time $t \in [0, T]$. Moreover, the quantity demanded by each demand market Q_j of kind k, at time $t \in [0, T]$, must be equal to the sum of all the commodity shipments of such kind from all the firms to that demand market, at the same time $t \in [0, T]$. As a consequence, for every $i = 1, \dots, m$, $j = 1, \dots, n$ and a.e. in $[0, T]$, the total production p_i

by the firm P_i, $i = 1, \dots, m$, and the total demand q_j of the demand market $Q_j, j = 1 \dots, n$, are given by

$$p_i(t) = \sum_{k=1}^{l} \sum_{j=1}^{n} x_{ij}^k(t), \quad \forall i = 1, \dots, m, \text{ a.e. in } [0, T],$$

$$q_j(t) = \sum_{k=1}^{l} \sum_{i=1}^{m} x_{ij}^k(t), \quad \forall j = 1, \dots, n, \text{ a.e. in } [0, T],$$

respectively.

We consider, then, the following convex, closed and bounded subset of the Hilbert space $L^2([0, T], \mathbb{R}^{[nml]})$ of feasible tensor mappings $\mathcal{X} \in L^2([0, T], \mathbb{R}^{[nml]})$:

$$\mathbb{K} = \Big\{ \mathcal{X} \in L^2([0, T], \mathbb{R}^{[nml]}) : \quad 0 \leq \underline{x}_{ij}^k(t) \leq x_{ij}^k(t) \leq \bar{x}_{ij}^k(t), \\ \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \forall k = 1, \dots, l \Big\}. \tag{23}$$

Moreover, let us introduce $f_i^k(t, \mathcal{X}(t))$, denoting the production cost of the firm P_i for each good of type k, at time $t \in [0, T]$, $i = 1, \dots, m$, $k = 1, \dots, l$, which depends upon the entire production pattern. Analogously, let us denote by $d_j^k(t, \mathcal{X}(t))$ the demand price for unity of the commodity of kind k for each demand market Q_j, $j = 1, \dots, n$, $k = 1, \dots, l$, assuming that depends upon the entire consumption pattern, at time $t \in [0, T]$. Finally, let $c_{ij}^k(t, \mathcal{X}(t))$ be the transaction cost, which includes the transportation cost associated with trading the commodity between the firm P_i and the demand market Q_j regarding the good of kind k, at time $t \in [0, T]$, $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, l$, and depending upon the entire shipment pattern.

In our model, the profit v_i of the firm P_i, $i = 1, \dots, m$, at time $t \in [0, T]$, is given by

$$v_i(t, \mathcal{X}(t)) = \sum_{k=1}^{l} \left[\sum_{j=1}^{n} d_j^k(t, \mathcal{X}(t)) x_{ij}^k(t) - f_i^k(t, \mathcal{X}(t)) - \sum_{j=1}^{n} c_{ij}^k(t, \mathcal{X}(t)) x_{ij}^k(t) \right],$$

namely the sum of the difference between the price that the demand markets are disposed to pay minus the production costs and the transportation costs.

The goal is to find a nonnegative commodity distribution tensor mapping $\mathcal{X}$ for which the m firms and the n demand markets will be in a state of equilibrium as defined below by means of a generalization of the Cournot–Nash equilibrium principle.

Definition 3.1: A feasible tensor mapping $\mathcal{X}^* \in \mathbb{K}$ is a general dynamic oligopolistic market equilibrium distribution if and only if, for each $i = 1, \dots, m$, it results

$$v_i(t, \mathcal{X}^*(t)) \geq v_i(t, X_i(t), \hat{\mathcal{X}}_{-i}^*(t)), \quad \forall \mathcal{X} \in \mathbb{K}, \text{ a.e. in } [0, T], \tag{24}$$

where $\hat{\mathcal{X}}_{-i}^*(t) = (X_1^*(t), \dots, X_{i-1}^*(t), X_{i+1}^*(t), \dots, X_m^*(t))$ and $X_i(t)$ is a slice of dimension nl.

Assuming that the profit function v is continuously differentiable, let us consider the tensor mapping

$$\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}^k} \right).$$

Moreover, we will say that $\nabla_D v$ satisfies Assumption (C) if the following conditions hold:

(i) $\nabla_D v$ is a Carathèodory function;
(ii) there exists $h \in L^2([0,T],\mathbb{R})$ such that

$$\|\nabla_D v(t,\mathcal{X}(t))\| \le h(t)\|\mathcal{X}(t)\|, \quad \forall \mathcal{X} \in L^2([0,T],\mathbb{R}^{[mnl]}).$$

Let us recall that the function v_i, $i = 1,\dots,m$, is said to be pseudoconcave with respect to the variable X_i, if

$$\left\langle \frac{\partial v_i}{\partial X_i}(X_1,\dots,X_i,\dots,X_m), X_i - Y_i \right\rangle \ge 0$$
$$\Rightarrow \; v_i(X_1,\dots,X_i,\dots,X_m) \ge v_i(X_1,\dots,Y_i,\dots,X_m).$$

Now, we can establish the following variational formulation.

Theorem 3.2: *Let us suppose that, for every firm P_i, the profit function $v_i(t,\mathcal{X}(t))$ is pseudoconcave with respect to the variable X_i, $i = 1,\dots,m$, a.e. in $[0,T]$, and continuously differentiable. Let us suppose that the tensor mapping $\nabla_D v$ satisfies Assumption (C). Then, $\mathcal{X}^* \in \mathbb{K}$ is a general dynamic Cournot–Nash equilibrium if and only if it satisfies the evolutionary tensor variational inequality*

$$\begin{aligned} &\ll -\nabla_D v(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \gg \\ &= -\int_0^T \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l \frac{\partial v_i(t,\mathcal{X}^*(t))}{\partial x_{ij}^k}(x_{ij}^k(t) - (x_{ij}^k)^*(t))dt \ge 0, \quad \forall \mathcal{X} \in \mathbb{K}. \end{aligned} \tag{25}$$

Proof: By Lemma 2.3, the evolutionary tensor variational inequality (25) is equivalent to

$$\langle -\nabla_D v(t,\mathcal{X}^*(t)), \mathcal{X}(t) - \mathcal{X}^*(t)\rangle \ge 0, \qquad \forall \mathcal{X}(t) \in \mathbb{K}(t), \text{ a.e. in } [0,T]. \tag{26}$$

To prove our claim, we assume first that $\mathcal{X}^*(t)$ satisfies the equilibrium condition (24), which is equivalent to

$$\langle -\nabla_D v_i(t,\mathcal{X}^*(t)), X_i(t) - X_i^*(t)\rangle \ge 0, \quad \forall \mathcal{X}(t) \in \mathbb{K}(t), \text{ a.e. in } [0,T],\ i = 1,\dots,m.$$

Since $\nabla_D v_i$ is a continuous function and $\mathcal{X},\mathcal{X}^* \in L^2([0,T],\mathbb{R}^{[nml]})$, it follows that $t \mapsto \langle -\nabla_D v_i(t,\mathcal{X}^*(t)), X_i(t) - X_i^*(t)\rangle \in L^2([0,T],\mathbb{R})$ and, moreover, it results

$$\int_0^T \langle -\nabla_D v_i(t,\mathcal{X}^*(t)), X_i(t) - X_i^*(t)\rangle\, dt \ge 0, \quad \forall \mathcal{X} \in \mathbb{K},\ i = 1,\dots,m.$$

Summing up the previous inequality over all $i = 1,\dots,m$, we obtain (25).

We assume now that $\mathcal{X}^* \in \mathbb{K}$ is a solution to (25) but is not an equilibrium distribution according to (24). As a consequence, there exist $I \subseteq [0, T]$, with $m(I) > 0$, $\bar{i} \in \{1, \dots, m\}$ and $\widehat{X}_{\bar{i}}$ such that

$$v_{\bar{i}}(t, \mathcal{X}^*(t)) < v_{\bar{i}}(t, \widehat{X}_{\bar{i}}(t), \mathcal{X}^*_{-\bar{i}}(t)), \quad \text{a.e. in } I.$$

By using the pseudoconcavity of the function $v_{\bar{i}}$, we deduce

$$\frac{\partial v_{\bar{i}}(t, \mathcal{X}(t))}{\partial X_{\bar{i}}}(X^*_{\bar{i}}(t) - \widehat{X}_{\bar{i}}(t)) < 0, \quad \text{a.e. in } I.$$

Choosing $\mathcal{X} \in \mathbb{K}$ such that

$$X_i(t) = \begin{cases} X^*_i(t), & \text{a.e. in } [0, T] \setminus I, \ \forall i = 1, \dots, m, \\ X^*_i(t), & \text{a.e. in } I, \ \text{for } i \neq \bar{i}, \\ \widehat{X}_{\bar{i}}(t), & \text{a.e. in } I, \ \text{for } i = \bar{i}, \end{cases}$$

in the left-hand side of inequality (25), we have

$$\ll -\nabla_D v(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \gg = \int_0^T \langle -\nabla_D v_{\bar{i}}(t, \mathcal{X}^*(t)), \widehat{X}_{\bar{i}}(t) - X^*_{\bar{i}}(t) \rangle dt < 0,$$

which is a contradiction. ■

The existence of the equilibrium solution follows by Theorem 2.8 taking into account that the feasible set $\mathbb{K}$ is a convex, closed and bounded subset of $L^2([0, T], \mathbb{R}^{[nml]})$. In particular it results:

Theorem 3.3: *Let us suppose that, for every firm P_i, the profit function $v_i(t, \mathcal{X}(t))$ is pseudoconcave with respect to the variable X_i, $i = 1, \dots, m$, a.e. in $[0, T]$, and continuously differentiable. Furthermore if $-\nabla_D v$ is a pseudomonotone tensor mapping satisfying Assumption (C). Then there exists at least a general dynamic Cournot–Nash equilibrium distribution $\mathcal{X}^* \in \mathbb{K}$.*

3.1. Continuity results for equilibrium distributions

It is also possible to establish conditions under which the general dynamic oligopolistic market equilibrium problem has continuous solutions with respect to the time variable. Before to prove such results, we show a preliminary lemma which states that the feasible set $\mathbb{K}$ satisfies the property of the Kuratowski set convergence.

Lemma 3.4: *Let $\underline{\mathcal{X}}, \overline{\mathcal{X}} \in C^0([0, T], \mathbb{R}^{[nml]})$ be nonnegative tensor functions, $t \in [0, T]$ and $\{t_r\}$ be a sequence such that $t_r \to t$, as $r \to +\infty$. Then, the set sequence*

$$\mathbb{K}(t_r) = \Big\{ \mathcal{X}(t_r) \in \mathbb{R}^{[nml]} : \ 0 \leq \underline{x}^k_{ij}(t_r) \leq x^k_{ij}(t_r) \leq \bar{x}^k_{ij}(t_r),$$

$$i = 1, \dots, m,\ j = 1, \dots, n,\ k = 1, \dots, l\Big\}, \quad \forall r \in \mathbb{N},$$

converges to

$$\mathbb{K}(t) = \Big\{\mathcal{X}(t) \in \mathbb{R}^{[nml]} : 0 \le \underline{x}_{ij}^k(t) \le x_{ij}^k(t) \le \bar{x}_{ij}^k(t),$$
$$i = 1, \dots, m,\ j = 1, \dots, n,\ k = 1, \dots, l\Big\},$$

in Kuratowski's sense.

Proof: Let us fix $t \in [0, T]$ and a sequence $\{t_r\} \subseteq [0, T]$ such that $t_r \to t$, as $r \to +\infty$. To reach the claim, it is enough to show that conditions (K1) and (K2) hold. Let $\mathcal{X}(t) \in \mathbb{K}(t)$ be fixed and let us consider the following sequence:

$$\mathcal{X}(t_r) = \underline{\mathcal{X}}(t_r) + \min\{\mathcal{X}(t) - \underline{\mathcal{X}}(t), \overline{\mathcal{X}}(t_r) - \underline{\mathcal{X}}(t_r)\}, \quad \forall r \in \mathbb{N}.$$

Let us note that $\mathcal{X}(t_r) \in \mathbb{K}(t_r)$, for every $r \in \mathbb{N}$. Indeed, being $\min\{\mathcal{X}(t) - \underline{\mathcal{X}}(t), \overline{\mathcal{X}}(t_r) - \underline{\mathcal{X}}(t_r)\} \ge 0$, for every $r \in \mathbb{N}$, we have $\mathcal{X}(t_r) \ge \underline{\mathcal{X}}(t_r)$, for every $r \in \mathbb{N}$. On the other hand, since $\min\{\mathcal{X}(t) - \underline{\mathcal{X}}(t), \overline{\mathcal{X}}(t_r) - \underline{\mathcal{X}}(t_r)\} \le \overline{\mathcal{X}}(t_r) - \underline{\mathcal{X}}(t_r)$, for every $r \in \mathbb{N}$, it follows $\mathcal{X}(t_r) \le \overline{\mathcal{X}}(t_r)$, for every $r \in \mathbb{N}$. Being $\underline{\mathcal{X}}(t) \le \mathcal{X}(t) \le \overline{\mathcal{X}}(t)$, in $[0, T]$, we deduce

$$\lim_{r \to +\infty} \mathcal{X}(t_r) = \lim_{r \to +\infty} \Big\{\underline{\mathcal{X}}(t_r) + \min\{\mathcal{X}(t) - \underline{\mathcal{X}}(t), \overline{\mathcal{X}}(t_r) - \underline{\mathcal{X}}(t_r)\}\Big\}$$
$$= \underline{\mathcal{X}}(t) + \min\{\mathcal{X}(t) - \underline{\mathcal{X}}(t), \overline{\mathcal{X}}(t) - \underline{\mathcal{X}}(t)\} = \mathcal{X}(t).$$

Then condition (K1) holds.

We prove now condition (K2). Let $\{\mathcal{X}(t_r)\}$ be a fixed sequence, with $\mathcal{X}(t_r) \in \mathbb{K}(t_r)$, for every $r \in \mathbb{N}$, such that $\mathcal{X}(t_r) \to \mathcal{X}(t)$, as $r \to +\infty$. It remains to show that $\mathcal{X}(t) \in \mathbb{K}(t)$. Since $\mathcal{X}(t_r) \in \mathbb{K}(t_r)$, for every $r \in \mathbb{N}$, i.e. $\underline{\mathcal{X}}(t_r) \le \mathcal{X}(t_r) \le \overline{\mathcal{X}}(t_r)$, for every $r \in \mathbb{N}$, passing to the limit as $r \to +\infty$, we obtain $\underline{\mathcal{X}}(t) \le \mathcal{X}(t) \le \overline{\mathcal{X}}(t)$. Hence, the claim is achieved. ■

Making use of Theorem 2.11 and Lemma 3.4, we obtain:

Theorem 3.5: *Let $\underline{\mathcal{X}}, \overline{\mathcal{X}} \in C^0([0, T], \mathbb{R}^{[nml]})$ be nonnegative tensor functions. Let us suppose that, for each firm P_i, the profit function $v_i(t, \mathcal{X}(t))$ is pseudoconcave with respect to the variable X_i, $i = 1, \dots, n$, belonging to $C^1([0, T] \times \mathbb{K}, \mathbb{R})$. Furthermore if $-\nabla_D v$ is a strongly monotone tensor function satisfying Assumption (C). Then the unique general dynamic Cournot–Nash equilibrium distribution $\mathcal{X}^* \in \mathbb{K}$ is continuous in $[0, T]$.*

Taking into account Theorem 2.12, Lemma 3.4 and the boundness of $\mathbb{K}$, we deduce:

Theorem 3.6: *Let $\underline{\mathcal{X}}, \overline{\mathcal{X}} \in C^0([0, T], \mathbb{R}^{[nml]})$ be nonnegative tensor functions. Let us suppose that, for each firm P_i, the profit function $v_i(t, \mathcal{X}(t))$ is pseudoconcave with respect to the variable X_i, $i = 1, \dots, m$, belonging to $C^1([0, T] \times \mathbb{K}, \mathbb{R})$. Furthermore if $-\nabla_D v$ is a strictly monotone tensor function satisfying Assumption (C). Then the unique general dynamic Cournot–Nash equilibrium distribution $\mathcal{X}^* \in \mathbb{K}$ is continuous in $[0, T]$.*

3.2. An example

Let us now consider an economic network consisting of two supply markets and two demand markets. Each firm produces two different kind of commodities. In Figure 1, the network is represented, precisely dashed and continuous lines depict the two kinds of commodities. We analyse the noncooperative behaviour of the firms in the time interval $[0, 2]$ computing its evolution in time. The feasible set is

$$\mathbb{K} = \Big\{ \mathcal{X} \in L^2([0,2], \mathbb{R}^{[8]}) : \; 2t \leq x_{ij}^k(t)$$
$$\leq 10t + 5, \quad i = 1, 2, \; j = 1, 2, \; k = 1, 2, \text{ a.e. in } [0, 2] \Big\}.$$

We consider the production cost function defined by

$$\begin{aligned}
f_1^1(t, \mathcal{X}(t)) &= t x_{11}^1(t) + 2x_{12}^1(t), \quad \text{a.e. in } [0,2],\\
f_1^2(t, \mathcal{X}(t)) &= x_{11}^2(t) + t x_{12}^2(t), \quad \text{a.e. in } [0,2],\\
f_2^1(t, \mathcal{X}(t)) &= 3x_{21}^1(t) + 2t x_{22}^1(t), \quad \text{a.e. in } [0,2],\\
f_2^2(t, \mathcal{X}(t)) &= (t - \tfrac{11}{2}) x_{21}^2(t) + x_{22}^2(t), \quad \text{a.e. in } [0,2],
\end{aligned}$$

and the demand price function given by

$$\begin{aligned}
d_1^1(t, \mathcal{X}(t)) &= t x_{11}^1(t) + x_{21}^1(t) + 2t - 1, \quad \text{a.e. in } [0,2],\\
d_1^2(t, \mathcal{X}(t)) &= \frac{1}{2} x_{11}^2(t) + x_{21}^2(t) + 3, \quad \text{a.e. in } [0,2],\\
d_2^1(t, \mathcal{X}(t)) &= x_{12}^1(t) + (t+1) x_{22}^1(t) + 2t, \quad \text{a.e. in } [0,2],\\
d_2^2(t, \mathcal{X}(t)) &= 2x_{12}^2(t) + t x_{22}^2(t) + 1, \quad \text{a.e. in } [0,2].
\end{aligned}$$

The cost transportation function is

$$\begin{aligned}
c_{11}^1(t, \mathcal{X}(t)) &= t x_{11}^1(t) + \frac{1}{2} x_{12}^1(t) + 3t, \quad \text{a.e. in } [0,2],\\
c_{21}^1(t, \mathcal{X}(t)) &= \frac{5}{2} x_{21}^1(t) + t x_{22}^1(t) + 4, \quad \text{a.e. in } [0,2],\\
c_{12}^1(t, \mathcal{X}(t)) &= \frac{3}{2} x_{12}^1(t) - \frac{1}{2} x_{11}^1(t) - 3, \quad \text{a.e. in } [0,2],\\
c_{22^1}(t, \mathcal{X}(t)) &= x_{22}^1(t) + x_{12}^2(t), \quad \text{a.e. in } [0,2],\\
c_{11}^2(t, \mathcal{X}(t)) &= x_{11}^2(t) - x_{12}^2(t), \quad \text{a.e. in } [0,2],\\
c_{21}^2(t, \mathcal{X}(t)) &= \frac{1}{2} x_{21}^2(t) + 2x_{22}^2(t) + \frac{1}{2} x_{11}^2(t) + 2, \quad \text{a.e. in } [0,2],\\
c_{12}^2(t, \mathcal{X}(t)) &= \frac{1}{2} x_{12}^2(t) + x_{11}^2(t) + \frac{t}{2}(t), \quad \text{a.e. in } [0,2],\\
c_{22}^2(t, \mathcal{X}(t)) &= x_{22}^2(t) + t x_{22}^2(t) - 2x_{21}^2(t), \quad \text{a.e. in } [0,2].
\end{aligned}$$

Then, the profit function becomes

$$\begin{aligned}
v_1(t,\mathcal{X}(t)) &= x_{11}^1(t)x_{21}^1(t) - (2t+1)x_{11}^1(t) - \frac{1}{2}(x_{11}^2(t))^2 + x_{11}^2(t)x_{21}^2(t) + 2x_{11}^2(t) \\
&\quad - \frac{1}{2}(x_{12}^1(t))^2 + (t+1)x_{12}^1(t)x_{22}^1(t) + (2t+1)x_{12}^1(t) + \frac{3}{2}(x_{12}^2(t))^2 \\
&\quad + tx_{12}^2(t)x_{22}^2(t) - x_{12}^2(t) - \frac{3}{2}tx_{12}^2(t), \quad \text{a.e. in } [0,2], \\
v_2(t,\mathcal{X}(t)) &= tx_{11}^1(t)x_{21}^1(t) - \frac{3}{4}(x_{21}^1(t))^2 + \left(2t+\frac{3}{2}\right)x_{21}^1(t) + \frac{1}{2}(x_{21}^2(t))^2 - (t+1)x_{21}^2(t) \\
&\quad + t(x_{22}^1(t))^2 + 2x_{12}^2(t)x_{22}^2(t) - (x_{22}^2(t))^2 - tx_{21}^1(t)x_{22}^1(t), \quad \text{a.e. in } [0,2].
\end{aligned}$$

Therefore the components of $\nabla_D v$ different from zero are given by

$$\begin{aligned}
&\frac{\partial v_1}{x_{11}^1}(t,\mathcal{X}(t)) = x_{21}^1(t) - 2t - 1, \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_1}{x_{12}^1}(t,\mathcal{X}(t)) = -x_{12}^1(t) + (t+1)x_{22}^1(t) + 2t + 1, \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_1}{x_{11}^2}(t,\mathcal{X}(t)) = -x_{11}^2(t) + x_{21}^2(t) + 2, \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_1}{x_{12}^2}(t,\mathcal{X}(t)) = 3x_{12}^2(t) + tx_{22}^2(t) - 1 - \frac{3}{2}t, \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_2}{x_{21}^1}(t,\mathcal{X}(t)) = tx_{11}^1(t) - \tfrac{3}{2}x_{21}^1(t) + 2t + \tfrac{3}{2} - tx_{22}^1(t), \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_2}{x_{22}^1}(t,\mathcal{X}(t)) = 2tx_{22}^1(t) - tx_{21}^1(t), \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_2}{x_{21}^2}(t,\mathcal{X}(t)) = x_{21}^2(t) - t - 1, \quad \text{a.e. in } [0,2], \\
&\frac{\partial v_2}{x_{22}^2}(t,\mathcal{X}(t)) = 2x_{12}^2(t) - 2x_{22}^2(t), \quad \text{a.e. in } [0,2].
\end{aligned}$$

By Theorem 3.2, the equilibrium distribution is a solution to the following evolutionary tensor variational inequality:

$$-\int_0^2 \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \frac{\partial v_i(t,\mathcal{X}^*(t))}{\partial x_{ij}^k}(x_{ij}^k(t) - (x_{ij}^k)^*(t))dt \geq 0, \quad \forall \mathcal{X} \in \mathbb{K}.$$

It can be verified that the tensor mapping $-\nabla_D v$ satisfies the assumptions of Theorem 3.5, thus the general dynamic oligopolistic market equilibrium example has a unique continuous equilibrium solution. We compute an approximate solution to the example by using the combination of a discretization procedure and the projection method presented in Section 2.3. Making use of Matlab computations to implement the algorithm, we obtain the equilibrium distribution curves represented in Figure 2.

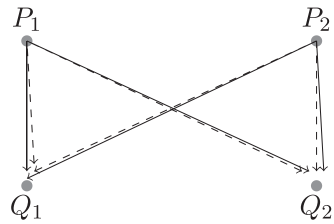

Figure 1. Network structure of the oligopoly.

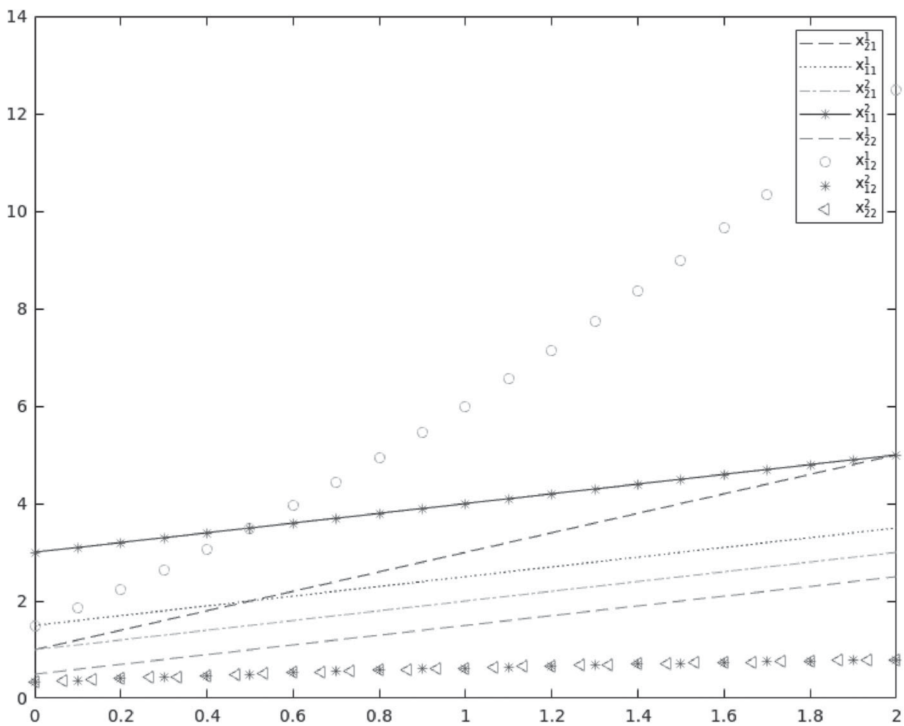

Figure 2. Computed equilibrium solution.

4. Concluding remarks

We introduced tensor variational inequalities in Hilbert spaces. Some existence and regularity results are proved. Furthermore a numerical discretization method combined with a projection one is presented to solve an evolutionary tensor variational inequality. The theoretical results are preliminary to analyse a general oligopolistic market equilibrium model in which each firm produces several commodities in a time interval. The firms act in a non-cooperative behaviour. Therefore, the equilibrium condition is established as an extension of the time-dependent Cournot–Nash principle. In addition, it is characterized by means of an evolutionary tensor variational inequality. Thanks to the variational formulation, the existence and the regularity of the time-dependent equilibrium distribution are obtained applying the results proved in the first part of the paper. At last a numerical example is discussed and solved with the approximate method presented.

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No potential conflict of interest was reported by the author(s).

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