CS389L: Automated Logical Reasoning

Lecture 17: SMT Solvers and the DPPL($\mathcal{T}$) Framework

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## Motivation

- In previous lectures, we looked at decision procedures for conjunctive formulas in various first-order theories
- This lecture: How to handle boolean structure when deciding satisfiability modulo theories
- In practice, cannot convert to DNF because causes exponential blow-up in formula size
- SMT (satisfiability modulo theory) solvers use clever techniques to handle boolean structure



## SMT solvers

- Key idea underlying SMT solvers: Combine theory solvers with SAT solvers
  - Theory solver: Decision procedure for checking satisfiability in conjunctive fragment
- SAT solver handles boolean structure, and theory solver handles theory-specific reasoning



## The Basic Idea

- To use SAT solver, we construct a propositional formula, called boolean abstraction, that overapproximates satisfiability
- If boolean abstraction is UNSAT, we are done $\Rightarrow$ also unsat modulo theory
- If boolean abstraction is SAT, doesn't necessarily mean original formula is SAT
  - Use theory solver to check if assignment returned by SAT solver is satisfiable modulo theory
- If not, add additional boolean constraints (called theory conflict clauses) to guide the search for an assignment that is satisfiable modulo theory



## Boolean Abstraction

- SMT formula in theory $\mathcal{T}$ formed according to CFG:

$$F := a_{\mathcal{T}} \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid \neg F$$

- For each SMT formula, define a bijective function $\mathcal{B}$, called boolean abstraction function, that maps SMT formula to overapproximate SAT formula
- Function $\mathcal{B}$ defined inductively as follows:

$$\begin{array}{rcl} \mathcal{B}(a_{\mathcal{T}}) & = & b \quad (b \text{ fresh}) \\ \mathcal{B}(F_1 \wedge F_2) & = & \mathcal{B}(F_1) \wedge \mathcal{B}(F_2) \\ \mathcal{B}(F_1 \vee F_2) & = & \mathcal{B}(F_1) \vee \mathcal{B}(F_2) \\ \mathcal{B}(\neg F) & = & \neg \mathcal{B}(F_1) \end{array}$$



## Example

- What is the boolean abstraction of this formula?

$$F: \quad x = z \wedge ((y = z \wedge x < z) \vee \neg(x = z))$$

-
- Boolean abstraction is also called boolean skeleton
- Since $\mathcal{B}$ is a bijective function, $\mathcal{B}^{-1}$ also exists
- What is $\mathcal{B}^{-1}(b_2 \vee \neg b_1)$?

## Boolean Abstraction as Overapproximation

- Observe: The boolean abstraction constructed this way overapproximates satisfiability of the formula
- Is this formula satisfiable?

$$F: \quad x = z \wedge ((y = z \wedge x < z) \vee \neg(x = z))$$

- Boolean abstraction: $\mathcal{B}(F) = b_1 \wedge ((b_2 \wedge b_3) \vee \neg b_1)$
- Is this satisfiable?
- What is a sat assignment?
- What is $\mathcal{B}^{-1}(A)$ ?
- Is $\mathcal{B}^{-1}(A)$ satisfiable?



## Need for Theory Conflict Clauses

- SAT solver may yield assignments that are not sat modulo $T$ because boolean abstraction is an over-approximation
- In this case, we need to learn theory conflict clauses
- Two different approaches for learning theory conflict clauses
  - Off-line (eager): Use SAT solver as black-box
  - On-line (lazy): Integrate theory solver into the CDCL loop
- First look into off-line version because it's easier



## SMT Solving Off-line Version

```
OfflineSMT(φ){
  ψ := B(φ)
  while(true){
    A := CDCL(φ)
    if(A = ⊥)  return UNSAT;
    res := TheorySolve(B^{-1}(A));
    if(res) return SAT;
     ψ := ψ ∧ ¬A
  }
}
```



## Example

- Consider example from before:

$$F: \quad x = z \wedge ((y = z \wedge x < z) \vee \neg(x = z))$$

- $\mathcal{B}(F): b_1 \wedge ((b_2 \wedge b_3) \vee \neg b_1)$
- Sat assignment to $\mathcal{B}(F)$ $A: b_1 \wedge b_2 \wedge b_3$
- $\mathcal{B}^{-1}(A)$ is unsat
- What is new boolean abstraction?

$$(b_1 \wedge ((b_2 \wedge b_3) \vee \neg b_1)) \wedge \neg(b_1 \wedge b_2 \wedge b_3)$$

- Is this formula SAT?



## Shortcoming of This Approach

- So far, we just add negation of current assignment as theory conflict clause
- Unfortunately, conflict clauses obtained this way are too weak
- Suppose $A$ is a conjunction of $100$ literals such that

$$\mathcal{B}^{-1}(A) = x = y \wedge x < y \wedge a_1 \wedge a_2 \wedge \ldots \wedge a_{98}$$

- Theory conflict clause $\neg A$ prevents exact same assignment
- But it doesn't prevent many other bad assignments involving $x = y \wedge x \neq y$ such as:

$$\mathcal{B}^{-1}(A) = x = y \wedge x < y \wedge a_1 \wedge a_2 \wedge \ldots \wedge \neg a_{98}$$

- In fact, there are $2^{98}$ unsat assignments containing $x = y \wedge x \neq y$ but $\neg A$ prevents only one of them!



## Improvement to Off-line SMT

- Rather than adding $\neg A$ as a conflict clause, better idea is to find an unsatisfiable core of $\mathcal{B}^{-1}(A)$
- Given a set $S$ of clauses, an unsat core of $S'$ is a subset $S'$ such that $S'$ is also unsat
- Ideally, we would like to find the minimal unsatisfiable core
- Minimal unsatisfiable core $C^*$ has property that if you drop any single atom of $C^*$, result is satisfiable
- What is a minimal unsat core of $x = y \wedge x < y \wedge a_1 \wedge a_2 \wedge \ldots \wedge a_{98}$? $x = y \wedge x < y$

## Computing Minimal Unsat Core

- How can we compute minimal unsat core of conjunctive $\mathcal{T}$ formula without modifying theory solver?
- Let $\phi$ be original unsatisfiable conjunct
- Drop one atom from $\phi$, call this $\phi'$
- If $\phi'$ is still unsat, $\phi := \phi'$
- Repeat this for every atom in $\phi$
- Clearly, resulting $\phi$ is minimal unsat core of original formula



## Example

- Let's compute minimal unsat core of

$$\phi: \ x = y \land f(x) + z = 5 \land f(x) \neq f(y) \land y \leq 3$$

- Drop $x = y$ from $\phi$. Is result unsat?
- Drop $f(x) + z = 5$. Is result unsat?
- New formula: $\phi: \ x = y \land f(x) \neq f(y) \land y \leq 3$
- Drop $f(x) \neq f(y)$. Is result unsat?
- Finally, drop $y \leq 3$. Is result unsat?
- So, minimal unsat core is $x = y \land f(x) \neq f(y)$



## SMT Improved Off-line Version

$$
\begin{array}{l}
\textbf{OfflineSMT}(\phi)\{ \\
\quad \psi := \mathcal{B}(\phi) \\
\quad \text{while(true)}\{ \\
\quad\quad A := \text{CDCL}(\phi) \\
\quad\quad \text{if}(A = \bot) \ \text{ return UNSAT;} \\
\quad\quad \text{res} := \text{TheorySolve}(\mathcal{B}^{-1}(A)); \\
\quad\quad \text{if(res) return SAT;} \\
\quad\quad \chi := \textbf{UnsatCore}(\mathcal{B}^{-1}(A)) \\
\quad\quad \psi := \psi \land \neg \mathcal{B}(\chi) \\
\quad \} \\
\}
\end{array}
$$



## Motivation for On-line SMT

- This strategy is much better than simple strategy where we add $\neg A$ as theory conflict clause.
- But still need to wait for full assignment from the SAT solver, which can be problematic
- Consider very large formula $F$ containing $x = y$ and $x < y$ with corresponding boolean variables $b_1$ and $b_2$
- As soon as sat solver makes assignment $b_1 = \top, b_2 = \top$, we are doomed because this is unsatisfiable in theory
- Thus, no need to continue with SAT solving after this bad partial assignment

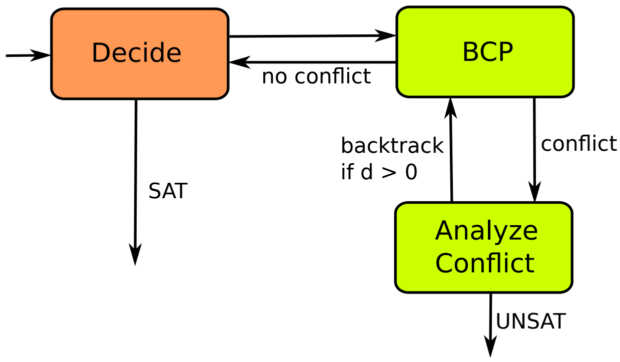


## On-line SMT

- Idea: Don't use SAT solver as "blackbox"
- Integrate theory solver right into the CDCL
- In other words, theory conflict clauses become another kind of conflict clause that SAT solvers already learn...

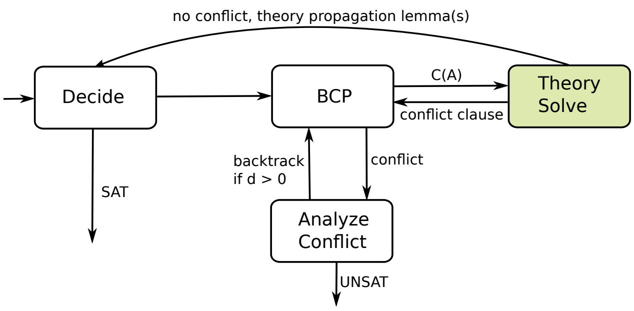


## DPLL-Based SAT Solver Architecture




- Idea: Integrate theory solver right into this SAT solving loop!

## DPLL($\mathcal{T}$) Framework




- Combination of DPLL-based SAT solver and decision procedure for conjunctive $\mathcal{T}$ formula called DPLL($\mathcal{T}$) framework

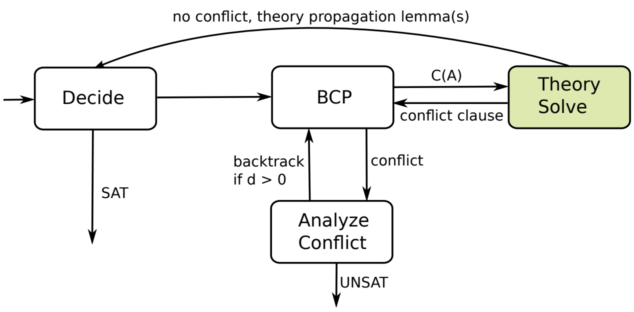


## DPLL($\mathcal{T}$) Framework

- Suppose SAT solver has made assignment in Decide step and performed BCP
- If no conflict detected, immediately invoke theory solver
- Specifically, suppose $A$ is current partial assignment to boolean abstraction
- Use theory solver to decide if $\mathcal{B}^{-1}(A)$ is unsat
- If $\mathcal{B}^{-1}(A)$ unsat, add theory conflict clause $\neg A$ to clause database
- Or better, add negation of unsat core of $A$ to clause database



## DPLL($\mathcal{T}$) Framework




- Add theory conflict clause and continue doing BCP, which will detect conflict
- As before, AnalyzeConflict decides what level to backtrack to



## Theory Propagation

- What we described so far is sufficient to solve SMT formula, but we can be even more clever!
- Suppose original formula contains literals $x = y, y = z, x < z$ with corresponding boolean variables $b_1, b_2, b_3$
- Suppose SAT solver makes partial assignment $b_1 : \top, b_2 : \top$
- In next Decide step, free to assign $b_3 : \top$ or $b_3 : \bot$
- But assignment $b_3 : \top$ is stupid b/c will lead to conflict in $\mathcal{T}$

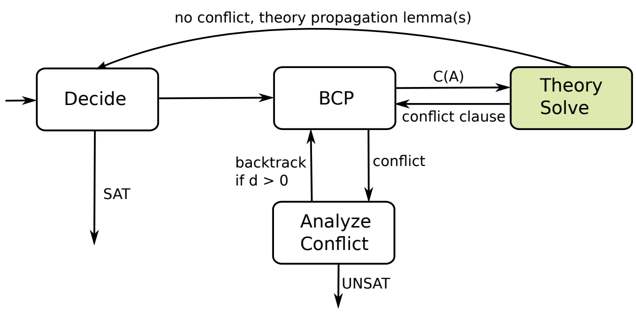


## Theory Propagation Lemma, cont

- Idea: Theory solver can communicate which literals are implied by current partial assignment
- In our example, $\neg x < z$ implied by current partial assignment $x = y \wedge y = z$
- Thus, can safely add $b_1 \wedge b_2 \rightarrow b_3$ to clause database
- These kinds of clauses implied by theory are called theory propagation lemmas



## DPLL($\mathcal{T}$) Framework




- Adding theory propagation lemmas prevents bad assignments to boolean abstraction

## Inferring Theory Propagation Lemmas

- How do we obtain theory propagation lemmas?
- Option #1: Treat theory solver as blackbox, query whether particular literal $a$ is implied by current partial assisgnment?
- Option #2: Modify theory solver so that it can figure out implied literals
- Second option is considered more efficient, but have to figure out how to do this for each different theory



## Which Theory Propagation Lemmas to Add

- Which theory propagation lemmas do we add?
- Option #1: Figure out and add all literals implied by current partial assignment; called exhaustive theory propagation
- Option #2: Only figure out literals "obviously" implied by current partial assignment
- Exhaustive theory propagation can be very expensive; second option considered preferable
- There isn't much of a science behind which literals are "obviously" implied
- Solvers use different strategies to obtain simple-to-find implications



## SMT Solvers Today

- All competitive SMT solvers today are based on the on-line version
- Many existing off-the-shelf SMT solvers: Z3, CVC3, Yices, MathSAT, etc.
- Lots of on-going research on SMT, esp. related to quantifier support
- Annual competition SMT-COMP between solvers; tools ranked in various categories



## Summary

- SMT solvers decide satisfiability in boolean combinations of different theories
- Instead of converting to DNF, they handle boolean structure using SAT solving technqiues
- Competitive solvers are based on DPLL($\mathcal{T}$) framework