CS389L: Automated Logical Reasoning

Lecture 5: Binary Decision Diagrams

Işıl Dillig

Motivation

- Previous lectures: How to determine satisfiability of propositional formulas
- Sometimes need to efficiently represent all solutions (i.e., satisfying assignments) to the formula
- Binary decision diagrams (BDDs): compact representation of all satisfying assignments of formula

Historical Context

- Invented by Randal Bryant from CMU; introduced in very influential 1986 paper
- BDDs have many applications: hardware and software verification, computer aided design of circuits, relational databases, ...
- Don Knuth: "One of the really fundamental data structures that came out in the last 25 years"

Binary Decision Trees

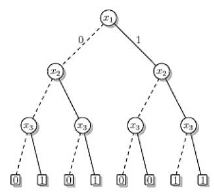
- Before talking about BDDs, let us first consider a simpler data structure called binary decision tree
- Binary decision tree: A tree data strcuture with two kinds of vertices, terminal and non-terminal
- Terminal vertices: boolean constants $\top(1)$ and $\bot(0)$
- Non-terminal vertex labeled v corresponds to boolean variable v in propositional formula

Binary Decision Trees, cont.

- Each non-terminal vertex has two successors (i.e., edges)
- Low successor of non-terminal vertex v (labeled with dashed edge) corresponds to assigning 0 to v
- High successor of vertex v (labeled with solid edge) corresponds to assigning 1 to v
- Each path from root to a terminal node corresponds to an interpretation for the formula
- Paths ending in 1 correspond to satisfying interpretations
- Paths ending in 0 correspond to falsifying interpretations

Example: Binary Decision Tree

- Binary decision tree for formula with variables x_1, x_2, x_3:



- Question: Is interpretation $x_1 = \bot, x_2 = \bot, x_3 = \top$ satisfying?
- Question: What about $x_1 = \top, x_2 = \bot, x_3 = \top$?
- This BDT is ordered: Any path from the root to a terminal contains variables in the same order (order: $x_1 < x_2 < x_3$)

Binary Decision Tree vs. Truth Table

- Binary decision tree encodes all satisfying assignments, but how does it compare to truth tables?
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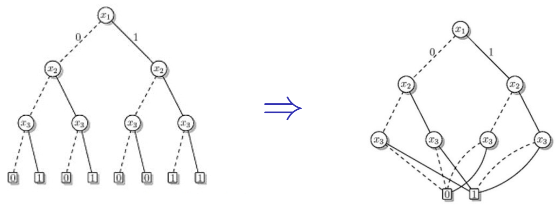
- Good news: Not as bad as it looks; there is a lot of redundancy! (e.g., different subparts are isomorphic)
- Idea: Merge redundant subparts and compress BDT into a much more space-efficient DAG representation!

Reduced Ordered Binary Decision Diagram

- To create compact representation, start with a ordered binary decision tree and apply a set of reduction rules
- The resulting data structure is called a reduced ordered binary decision diagram (ROBDD)
- When we talk about BDDs, we really mean ROBDDs (and so does everyone else)

Reduction Rule #1

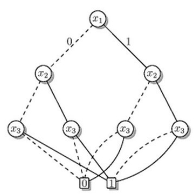
Reduction rule #1: Merge all terminal nodes 1 into one node, and merge all terminal nodes 0 into one node.



Reduction Rule #2

- Reduction rule #2: Merge isomorphic subgraphs
- Two subgraphs are isomorphic if:
  1. Their root represents the same variable
  2. The subgraphs rooted at their low successors are isomorphic
  3. The subgraphs rooted at their high successors are also isomorphic

Example: Merging Isomorphic Subtrees

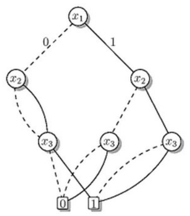


- Which subgraphs are isomorphic?

Reduction Rule #3

- Reduction rule #3: Remove redundant nodes.
- Node v is redundant if its low and high successors are the same.
- Why?
- Eliminating redundant node v: Remove v from the graph, and redirect incoming edge to v to v's children

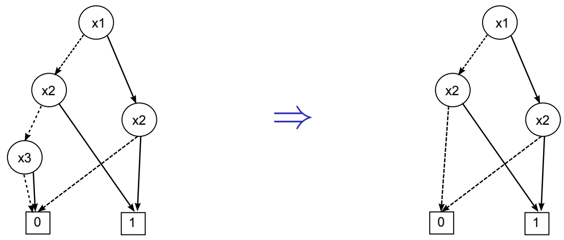
Example: Removing Redundant Nodes



- Which nodes are redundant?

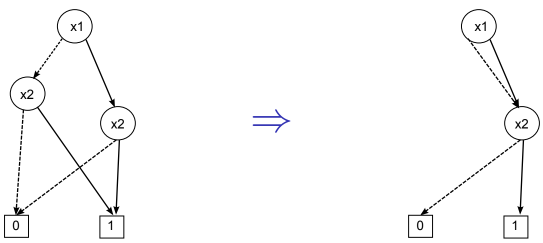
Exposing New Reductions

- Question: Can removing redundant nodes expose new isomorphic subtrees? (i.e., do we need to apply reduction 2 again after reduction 3?)
- Example: Are there any isomorphic subparts in this graph?



Exposing New Reductions, cont.

- Question: Can merging isomorphic subgraphs expose new redundant nodes? (i.e., do we need to apply reduction 3 again after reduction 2?)
- Example: Are there any redundant nodes in this graph?



Applying Reductions

- As examples illustrate, need to apply reduction rules until fixed point
- Resulting data structure after exhaustive application of reduction rules is a ROBDD.
- ROBDD is more space efficient compared to the binary decision tree because it eliminates redundancies.

Building ROBDDs

- Starting with BDT and applying reduction rule useful way to understand BDD invariants
- But no one builds BDDs this way. Why?
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- Idea: Build the ROBDD for a formula directly without building the binary decision tree!

Building BDDs directly from Formulas

- Consider a formula ϕ of the form $\phi_1 \star \phi_2$ where $\star$ is any boolean connective
- To construct the ROBDD for ϕ, first recursively construct the ROBDDs for ϕ_1 and ϕ_2
- Then, combine BDDs for ϕ_1 and ϕ_2 to form the BDD for ϕ
- To combine BDDs for ϕ_1 and ϕ_2, will use a technique called Shannon's decomposition

Cofactoring and Shannon's Decomposition

- Shannon's decomposition involves an operation called cofactoring.
- Cofactoring a boolean formula restricts the formula to a particular value of a variable.
- Example: What is the resulting formula when we restrict x_1 to $\top$ in $x_1 \land x_2$?
- The positive cofactor $\phi \downarrow x_i$ is the resulting formula when x_i is replaced by $\top$
- The negative cofactor $\phi \downarrow \neg x_i$ is ϕ with x_i replaced by $\bot$
- Example: What is $(x_1 \lor (\neg x_2 \land x_3)) \downarrow \neg x_2$?

Cofactoring Using BDDs

- If we have a BDD for ϕ, it is easy to build BDD for positive and negative cofactors of ϕ with respect to x
- Given BDD for ϕ, how do we build BDD for $\phi \downarrow x$?
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- How do we build BDD for $\phi \downarrow \neg x$?
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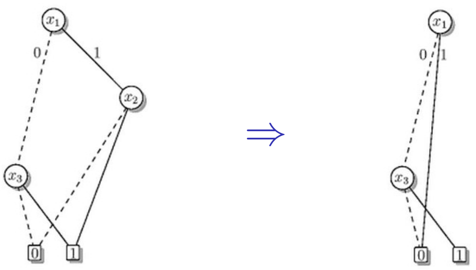
Cofactoring Example

What is the BDD representing $\phi \downarrow \neg x_2$?



Shannon's Decomposition

- Can now define Shannon's decomposition using cofactors.
- Shannon's decomposition:

$$\phi \equiv (x \land \phi \downarrow x) \lor (\neg x \land \phi \downarrow \neg x)$$

- Basically a case analysis on x's truth value
- If x is $\top$, then the positive cofactor must be true
- If x is $\bot$, then the negative cofactor must be true

Using Shannon's Decomposition to Build BDD

- Now, we can directly build BDD for a formula $\phi_1 \star \phi_2$ using Shannon's decomposition
- First, build BDD for subformulas ϕ_1 and ϕ_2
- Then, use recursive procedure called Apply to build BDD for ϕ from BDDs for ϕ_1 and ϕ_2

The Base Case

- Suppose that the BDDs for ϕ_1 and ϕ_2 have root nodes x and x' and both are boolean constants
- Then, what is the BDD for $\phi_1 \star \phi_2$?
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Recursive Step

- Given root nodes x, x', suppose x precedes x' according to variable order
- First do a case analysis on x using Shannon's decomposition
- Using Shannon's decomp., what is $\phi_1 \star \phi_2$ equivalent to?
- Cofactoring distributes over connectives, so rewrite as:

$$(x \wedge (\phi_1 \downarrow x \;\star\; \phi_2 \downarrow x) \vee (\neg x \wedge (\phi_1 \downarrow \neg x \;\star\; \phi_2 \downarrow \neg x))$$

Recursive step, continued

- Thus, we need to build BDD for:

$$(x \wedge (\phi_1 \downarrow x \;\star\; \phi_2 \downarrow x) \vee (\neg x \wedge (\phi_1 \downarrow \neg x \;\star\; \phi_2 \downarrow \neg x))$$

- We know how to build BDDs for $\phi_i \downarrow x$ and $\phi_i \downarrow \neg x$
- How do we build BDDs for $(\phi_1 \downarrow x \;\star\; \phi_2 \downarrow x)$ and $(\phi_1 \downarrow \neg x \;\star\; \phi_2 \downarrow \neg x)$?
-
- What is the progress/termination argument?
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Recursive step, continued

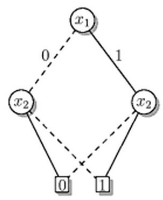
- Now, assume we have the BDDs for $(\phi_1 \downarrow x \;\star\; \phi_2 \downarrow x)$ and $(\phi_1 \downarrow \neg x \;\star\; \phi_2 \downarrow \neg x)$.
- Using these, how do we build BDD for the whole formula?

$$(x \wedge (\phi_1 \downarrow x \;\star\; \phi_2 \downarrow x) \vee (\neg x \wedge (\phi_1 \downarrow \neg x \;\star\; \phi_2 \downarrow \neg x))$$

  1. Make a new non-terminal node for variable x
  2. The low successor of this node is the BDD for the negative cofactor $(\phi_1 \downarrow \neg x \;\star\; \phi_2 \downarrow \neg x)$
  3. The high successor of this node is the BDD for the positive cofactor $(\phi_1 \downarrow x \;\star\; \phi_2 \downarrow x)$
- Note: We still need to apply the three reduction rules to ensure no redundancies.

An Example

- Consider formula $\phi_1 \vee \phi_2$ where ϕ_1 is $x_1 \leftrightarrow x_2$, and ϕ_2 is $\neg x_2$
- Suppose x_1 preceeds x_2 in variable order
- Assume we have BDDs for ϕ_1 and ϕ_2:



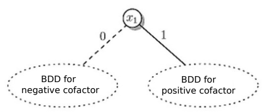
- We want to use Apply to compute BDD for $\phi_1 \vee \phi_2$

Example, continued

- Since x_1 preceeds x_2 in variable order, what do we perform case split on?
- How do we decompose $\phi_1 \vee \phi_2$ using Shannon's decomposition?
- Thus, we recursively compute BDDs for $(\phi_1 \downarrow \neg x_1 \;\vee\; \phi_2 \downarrow \neg x_1)$ and $(\phi_1 \downarrow x_1 \;\vee\; \phi_2 \downarrow x_1)$.

Example, continued

- Assuming we computed the BDDs for negative and positive cofactors, BDD for $\phi_1 \vee \phi_2$ looks like:

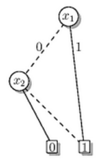


- Here, the positive cofactor is just 1
- Negative cofactor is:

- What is the final BDD?

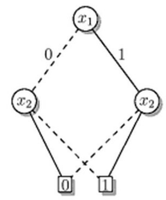
Example, continued

Final BDD:



Example 2

- Give the BDD representing the negation of the following BDD:



Nice BDD Properties: Canonicity

- BDDs are a canonical representation of boolean formulas.
- Canonical means two equivalent formulas have same representation
- Thus, if we construct BDDs for two equivalent formulas using same variable order, resulting BDDs are the same!
- Are any of the normal forms we talked about (NNF, CNF, DNF) canonical?
- Example:

Consequences of Canonicity

- Corollary 1: Given BDDs for ϕ_1 and ϕ_2 constructed with same variable order, equivalence of ϕ_1 and ϕ_2 just syntactic check
- Corollary 2: Given BDD for formula ϕ, ϕ is unsatisfiable if and only if its BDD respresentation is the boolean constant 0.
- How does this corollary follow from canonicity?
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- Corollary 3: Given BDD for formula ϕ, ϕ is valid if and only if its BDD respresentation is the boolean constant 1.

BDD Size and Variable Order

- Another important BDD property: Size of a BDD for a given formula ϕ is very sensitive to variable order!
- For some variable orders, the size of the BDD may be only polynomial in the number of variables
- For some other variable orders, the size of the BDD for same formula may be exponential.
- Furthermore, there are boolean formulas for which any variable order causes an exponential blow-up

Choosing Variable Order

- Since size of BDD is so sensitive to variable order, we would like to construct BDD using a good variable order
- Unfortunately, NP-complete to decide whether a given order is optimal
- Typically, heuristics are used to predict good variable order
- Heuristics for finding good variable order can be either static (i.e., determined up-front) or dynamic (i.e., change as BDD operations proceed)
- Dynamic orders typically yield more compact BDDs, but slower

Summary

- BDDs can be used to compactly represent all satisfying assignments to a boolean formula
- Pros:
  - + Canonical representation $\Rightarrow$ checking equivalence, validity, satisfiability constant time operations once BDD is built
  - + If we have good variable order, can yield a compact representation of all satisfying assignments of formula
- Cons:
  - - Compactness very sensitive to variable order
  - - Can cause an exponential blow-up in the representation of the formula